Worksheet 8

Math 607, Connections and Characteristic Classes

Monday, February 15, 2021

1. Let α be a k-form on $X \times \mathbb{R}$. For each $t \in \mathbb{R}$ we get a form α_t on X by restricting to $X \times \{t\}$. Convince yourselves that if α is closed then $\alpha_1 - \alpha_0$ is exact.

2. Let E be a vector bundle on X, and let ∇_0 and ∇_1 be two connection on E. I claim that $c_k(\nabla_0) - c_k(\nabla_1)$ is exact. Read the following argument and discuss any details you're not sure about:

   Consider $X \times \mathbb{R}$, and let

   $$p\colon X \times \mathbb{R} \to X$$

   be the projection. On the pull-back bundle p^*E we get two pull-back connections $p^*\nabla_0$ and $p^*\nabla_1$. Their weighted average

   $$\nabla := tp^*\nabla_1 + (1-t)p^*\nabla_0$$

   is again a connection on p^*E. The Chern form $c_k(\nabla)$ is a closed $2k$-forms on $X \times \mathbb{R}$, and its restriction to $t = 0$ or $t = 1$ is $c_k(\nabla_0)$ or $c_k(\nabla_1)$. Thus the $c_k(\nabla_1) - c_k(\nabla_0)$ is exact.