Claim 1: Let $S^2 = \{ x \in \mathbb{R}^2 \mid |x| = 1 \}$ and let $f: S^2 \to \mathbb{R}$ be continuous.

Then $\exists x \in S^2$ with $f(x) = f(-x)$.

Idea: consider $g(x) = f(x) - f(-x)$

$g: S^2 \to \mathbb{R}$

seek $x$ with $g(x) = 0$

suppose $g(x) > 0$ for some $x$

then $g(-x) = -g(x) < 0$

somewhere in between, must cross zero by the intermediate value theorem.

on worksheet: write this nicely.

Claim 2: Let $f: S^2 \to \mathbb{R}^2$ be continuous.

Then $\exists x \in S^2$ with $f(x) = f(-x)$.
Idea: Again consider \( g(x) = f(x) + f(-x) \)

Suppose \( g \) is never zero, produce a contradiction.

Let \( y : [0,1] \to S^2 \) parametrize the equator.

\[ \begin{array}{cc}
(0,0) & 1 \\
0 & 0
\end{array} \]

Let \( h = g \circ y : [0,1] \to \mathbb{R}^2 \setminus \{0\} \)

\[ h(0) = h(1) \] so \( h \) winds around the origin some \( n \) of times.

Claim it's an odd \( n \).

\[ \frac{1}{2} \] winds \( \frac{n}{2} \) times, rotated \( \phi_0 \).

\[ h(\frac{1}{2}) = -h(0) \]

\[ \frac{1}{2} \] winds \( n+1 \) times in total.
Now deform $\gamma$: let $\gamma_s$ parametrize the parallel $z = s$, $0 \leq s \leq 1$.

Let $h_s = g \circ \gamma_s$. Winding number is constant as $s$ varies.

But $\gamma_1$ is the constant map $\gamma_1(t) = (0, 0, 1)$, so $h_1$ is constant, so winding # of $h_1 = 0$.

Claim 3: $f : S^2 \to \mathbb{R}^2$ constant then $g \in S^2$ s.t. $f(g) = f(-g)$.

False: $g$ could be the natural embedding $S^2 \to \mathbb{R}^2$. 