\( \ker = \text{normal subgroup} \)  
\( \text{generated by abba} \)

\[ \begin{align*}
\mathcal{H}_a \times \mathcal{H}_b & \longrightarrow \mathcal{H}_c \times \mathcal{H}_d \\
G = \langle a, b \mid bbcb = 1 \rangle & \longrightarrow \mathcal{H} = \langle c, d \mid cdcd^{-1} = 1 \rangle \\
\text{check: abba} & \longrightarrow \end{align*} \]

and the other way

Last Wednesday: \( \pi_1(S^n) = 0 \)  
\( n \geq 2 \)  
Cor: \( \mathbb{R} \) homeomorphism \( f: \mathbb{R}^2 \to \mathbb{R}^n \)  
\( n \geq 3 \)  
because it would induce a homeo.

\[ \mathbb{R}^2 \setminus 0 \longrightarrow \mathbb{R}^n \setminus f(0) \]

and applying \( \pi_1 \),  
\[ \mathbb{H} \longrightarrow 0 \]  
can't be an iso.
Friday's worksheet: \( \pi_1(S^1 \cup S^1) = \mathbb{Z} \times \mathbb{Z} \)

\( X \cup U \) is contractible

apply van Kampen's thm.

Notice: works for any \( \pi_1(X \cup Y) = \pi_1(X) \ast \pi_1(Y) \)

as long as basepoint \( x_0 \in X \)

has a "good" neighborhood


has a "good" neighborhood

and sim. with \( y_0 \in Y \).

Happens in any decent example, but not in \( x_0 \) is not good

\( (\mathbb{R} \times \mathbb{R}) \cup \mathbb{R} \times \{0\} \)
Set-up for van Kampen: \( X = U \cup V \)

\[ \begin{array}{c}
U \cup V \xrightarrow{\text{inj}} U \\
\downarrow \quad \downarrow \quad \downarrow \\
V \rightarrow X \\
\end{array} \]

\[ \pi_n(U) \times \pi_n(V) \xrightarrow{i_+ \times i_-} \pi_n(X) \]

Let \( N \) be a normal subgroup generated by \( k_+ [x] \times l_+ [y]^{-1} \)

for \( [x] \in \pi_n(U) \)

then \( i_+ \times i_- \) is surjective

\[ \ker = N \]

so \( \pi_n(X) = \frac{\pi_n(U) \times \pi_n(V)}{N} \)

\[ \pi_1([x]) = l_+ [y] \]

\( \forall [y] \in \pi_1(U \cup V) \)
\begin{enumerate}
\item \( y \) is surjective.
\end{enumerate}

\begin{align*}
\text{Let } y & : [0,1] \to X, \quad y(0) = y(1) = x_0 \\
\text{by usual compactness argument, & there exists a partition of } [0,1] \\
&: 0 = s_0 < s_1 < \ldots < s_n = 1 \\
\text{s.t. } y([s_{i-1}, s_i]) & \subseteq U \text{ or } V \quad \forall i = 1, \ldots, n \\
\text{and } y(s_i) & \in U \cup V \quad \forall i \\
\text{Choose paths } & \alpha_i : x_0 \to y(s_i) \quad \forall i \\
\text{put } & y_i = \alpha_{i-1} \circ y|[s_{i-1}, s_i] \cdot x_i \quad \forall i \\
\text{then } & [y_i] \in \mathcal{T}_i(U) \text{ or } \mathcal{T}_i(V) \quad \text{depending on } i, \\
\text{and } & [y] = \iota_0 [y_0] \cdot \iota_1 [y_1] \cdot \ldots \cdot \iota_n [y_n].
\end{align*}
2. injectivity: \( \ker (i_1^* + \cdots + i_n^*) = N \)

Suppose \( \{f_1\} + \{f_2\} + \cdots + \{f_m\} \)

\( \in \pi_1(U) \neq \pi_1(V) \)

And \( \{g_1\} + \{g_2\} + \cdots + \{g_n\} \)

Another one,

And same image in \( \pi_1(X) \).

Want: \( \text{Sum of } \pi_1(U) + \pi_1(V) \)/\(N\)

Can assume \( M = n \)

Let \( F: I \times I \rightarrow X \)

Be a homotopy from \( f_1 \cdots f_n \) to \( g_1 \cdots g_n \)
By usual compactness argument, there exists a number \( M > 0 \) such that if we subdivide \( \mathbb{R} \times \mathbb{R} \) into \( M^2 \) squares, then \( F(\text{each arc}) \) is in \( U \) or \( V \).

Assume \( n = 1 \) (picture: \( M = n \))

Label vertices \( v_{11}, v_{12}, \ldots \)

Choose a path \( x_{ij} \) from \( x_0 \) to \( F(v_{ij}) \)

If \( F(v_{ij}) \) is in \( U \), make \( x_{ij} \) stay in \( U \)

--- --- U --- --- --- --- --- V
Let $\beta_{11}, \beta_{12}, \ldots, \beta_{n1}, \beta_{n2} = F$ restrict to horizontal segment.

Let $y_{11}, y_{12}, \ldots$ = $F$ restrict to vertical segment.

First row is $f_1, f_2, \ldots, f_n$.

Second row gives

$$
(\beta_{11} x_{11}^{-1})(\alpha_{11} \beta_{12} x_{12}^{-1}) \cdots (\alpha_{n1}^{-1} \beta_{nn}^{-1})
$$

both in $\pi_1(U) \times \pi_1(U)$

Want: same in $\pi_1(U) \times \pi_1(U)/N$

$$
[f_1] \cdot [f_2] \cdots [f_n]
$$

$$
= [\beta_{11} x_{11}^{-1}] [\alpha_{11} y_{11}^{-1}] + [f_2] + \cdots + [f_n]
$$

$$
= [\beta_{11} x_{11}^{-1}] [\alpha_{11} y_{11}^{-1}] + [f_2] + \cdots + [f_n] \in \pi_1(U)
$$
So we can cancel them here, we've modded out $N$.

$$= [p_{11} x_{11}] + [a_{11} p_{12} x_{12}] + [a_{12} x_{12}] + \cdots + (p_{1n})$$

Keep going square by square, end up with $g_1 = S_1$. 