Attaching Discs

Let $X$ be a space,

$f: S' \to X$

we can attach $D^2$ along $f$:

$X \cup D^2
\begin{array}{c}
p \in D^2 = S' \sim f(p) \in X
\end{array}$

Example: $X = S' \cup S'$

attach a disc along $a \sim \text{get torus}$

attach a disc along $a \sim \text{get a Klein bottle}$

From van Kampen's theorem:

$I_1(X \cup D^2, f(1)) = I_1(X, f(1))/\langle f \rangle$

Proof:
Let $U =$ interior of disc

$V = \bigcup f$ collar of the disc

$\pi_1(U) = \mathbb{Z}$

$V$ deformation retract onto $X$

$\pi_1(U) = \pi_1(X)$

$U \cup V \cong S^1$

$\pi_1(U \cup V) = \mathbb{Z}$

$\pi_1(U \cup V) \rightarrow \pi_1(U)$

$\ell \in \mathbb{Z} \rightarrow 1$

$\pi_1(U \cup V) \rightarrow \pi_1(V) = \pi_1(X)$

$\ell \in \mathbb{Z} \rightarrow [f]$}

$\pi_1 \left( \mathbb{R}P^2 \right) = \pi_1(X) \times 1 \twoheadrightarrow \pi_1(X) / \langle [f] \rangle$

$\left\langle \ell \right\rangle = 1$

Examples: $\pi_1 \left( \text{torus} \right) = \left\langle a, b \mid a 6a^{-1} b^{-1} = 1 \right\rangle$

$\pi_1 \left( \text{Klein bottle} \right) = \left\langle a, b \mid a 6a^{-1} b^{-1} = 1 \right\rangle$
Similarly, given a map \( f : S^{n-1} \to X, n \geq 3 \)
\[
\text{define } X_f \cup D^n = X \cup D^n \quad \rho \in \partial D^n = S^{n-1} \quad f(\rho) \in X
\]

Some van Kampen argument
\[
\pi_1(X_f \cup D^n) = \pi_1(X)
\]

because now \( U \cup V = S^{n-1} \)
\[
\pi_1(U \cup V) = 0 \quad \text{b/c } \quad n \geq 3.
\]

A CW complex:

Start with some points.
attach some \( D^1 \)'s
attach some \( D^2 \)'s
etc.

Finite dimensional CW complex: stop at some point.
\( \infty \)-dim CW complex: worry about topology on the direct limit.

all our nice spaces are homeomorphic to CW complexes
or at least homotopy equivalent.
of course they admit many cell structures:

\[ S^2 = D^2 \text{ attached to a point} \]

but also

or