Induced Maps + Homology Invariance

Last time:

- \( C^n_\partial(x) = \text{formal linear combo of maps } \sigma : \mathbb{S}^n \to X \)

- Differentials \( \partial : C^{n+1}(X) \to C^n_\partial(X) \)
  - \( \partial \circ \partial = 0 \)
  - \( \text{Im } \partial_{n+1} \subset \ker \partial_n \)
  - \( \text{boundaries } \subset \text{cycles} \)

- \( H^n(X) = \ker \partial_n / \text{Im } \partial_{n+1} \) tends to be fin. gen.

- Elements of \( H^n \) are represented by \( n \)-cycles

- Two cycles represent the same class in \( H^n \) if they differ by a boundary. ("homologous")

- \( [c] = [d] \)
- \( [c] = [b] + [c] \)
First calculation:

Prop. If \( X \) is path-connected, then \( H_0(X) = \mathbb{Z} \)

Proof: View a point \( x_0 \in X \) as a 0-chain

Then \( \partial x_0 = 0 \) so \( [x_0] \in H_0(X) \)

If \( x_1 \in X \) is another point,
choose a path \( y \) from \( x_0 \) to \( x_1 \)
View \( y \) as a 1-chain
Then \( \partial y = x_1 - x_0 \)

so \( [x_1] = [x_0] \) in \( H_0 \).

a gen. elt. of \( H_0 \) is \( a_0 [x_0] + a_1 [x_1] + \cdots + a_k [x_k] \)

But \( \partial (\sum a_i [x_i]) = \sum (\sum a_i) \partial [x_i] = \sum a_i \partial [x_i] \)

so \( H_0(X) = \mathbb{Z} \).

Prop. \( H_n(\text{point}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases} \)

Proof. \( \sum \) only one map \( \sigma : \Delta^n \to \text{point} \)

\[ \sum \sigma_i = \sum (-1)^i \sigma_i \circ \varphi_i \]

\[ \sum \sigma_i = \begin{cases} \sigma_i & \text{if } \sigma_i \text{ even} \\ 0 & \text{if } \sigma_i \text{ odd} \end{cases} \]
So \( \cdots \to C_3 \to C_2 \to C_1 \to C_0 \to 0 \) is

\[
\begin{align*}
\tilde{H}_0 &= \mathbb{R} / 0 = \mathbb{R} \\
\tilde{H}_1 &= \mathbb{R} / \mathbb{R} = 0 \\
\tilde{H}_2 &= 0 / 0 = 0 \\
\tilde{H}_3 &= \mathbb{R} / \mathbb{R} = 0 \\
e &tc. \quad \square
\end{align*}
\]

Suppose \( f : X \to Y \).

Define \( f_* : C_n(X) \to C_n(Y) \) and extend \( \sigma \mapsto f_\# \sigma \) linearly.

**Lemma** \( \exists \tilde{f}_* \sigma = f_* \tilde{\sigma} \) for all \( \sigma : \Delta^n \to X \) hence for all chains.

**Proof** \( \tilde{f}_* \sigma = \tilde{f} (f_\# \sigma) = \sum_{i=0}^n (-1)^i \tilde{f}_* (f_\# \sigma \circ \varphi_i) = \tilde{f} \left( \sum_{i=0}^n (-1)^i \tilde{f}_* (f_\# \sigma \circ \varphi_i) \right) \) \( \square \)
As a diagram

\[ \cdots \rightarrow C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\partial} C_{n-1}(X) \rightarrow \cdots \]

\[ \xrightarrow{f_*} \quad \xrightarrow{f_*} \quad \xrightarrow{f_*} \]

\[ \cdots \rightarrow C_{n+1}(Y) \xrightarrow{\partial} C_n(Y) \xrightarrow{\partial} C_{n-1}(Y) \rightarrow \cdots \]

Thus, \( f_*(\mathbb{Z}_n(X)) \subset \mathbb{Z}_n(Y) \)

if \( \partial c = 0 \) then \( \partial(f_*(c)) = f_*(\partial c) = 0 \)

and \( f_*(\mathbb{Z}_n(X)) \subset \mathbb{Z}_n(Y) \)

so we get a well-defined map

\[ f_*: H_n(X) \rightarrow H_n(Y) \]

Easy to check: \( (g \circ f)_* = g_* \circ f_* \)

\[ 1_* = 1 \]

Thus homeomorphic spaces have isomorphic \( H_n \)s.

Want: homotopy invariance.
Prop: if \( f \approx g : X \to Y \) then \( f_{\#} = g_{\#} : \text{H}_n(X) \to \text{H}_n(Y) \)

**Pf.** Let \( F: X \times I \to Y \) be a homotopy from \( f \) to \( g \).

Define \( P : C_n(X) \to C_{n+1}(Y) \) as follows:

Given \( \sigma : \Delta^n \to X \), consider

\[
\Delta^{n+1} \to \Delta^n \times I \xrightarrow{\sigma \times 1} X \times I \xrightarrow{F} Y
\]

Define \( P(\sigma) = \sum (-1)^i F \circ (\sigma \times 1) \circ p_i \) and extend linearly.

I claim that \( 2P + D_2 = g_{\#} - f_{\#} \) for \( \sigma : \Delta^n \to X \).
\[ \mathcal{P}_0 \text{ is sides, top, bottom} \]

\[ \mathcal{P}_1 \text{ is just the sides} \]

\[ g_0 \sigma \text{ is just the top} \]

\[ f_\ast \sigma \text{ is bottom, opp. orientation} \]

Worksheet: Conclude that \( f_\ast = g_0 \).