Final Exam

Due Friday, December 11, 2020

You may refer to Hatcher's book, the lecture notes, and the homework problems and solutions, and you may quote any results proved there. But do not consult your colleagues or the internet.

1. Some homological algebra.

   (a) The *cokernel* of a homomorphism $f\colon A \to B$ is defined to be $B/\operatorname{im} f$. Show that the quotient map $q\colon B \to \operatorname{coker} f$ enjoys the following universal property: we have $q \circ f = 0$, and for every map $\varphi\colon B \to Y$ with $\varphi \circ f = 0$, there is a unique map $\psi\colon \operatorname{coker} f \to Y$ such that $\psi \circ q = \varphi$.

   $$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "q"] \arrow[dr, "\varphi"'] & \operatorname{coker} f \arrow[d, dotted, "\psi"] \\ & & Y \end{tikzcd}$$

   (b) State, but do not prove, the dual universal property for the inclusion $i\colon \ker f \to A$.

   Hint: It involves maps into A rather than maps out of B.

   (c) Show that if

   $$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

   is an exact sequence, then $\ker h = \operatorname{im} g \cong \operatorname{coker} f$.

   (d) Show that if

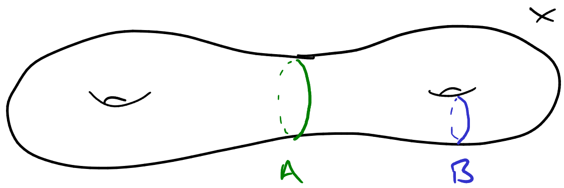
   $$0 \to A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \to 0$$

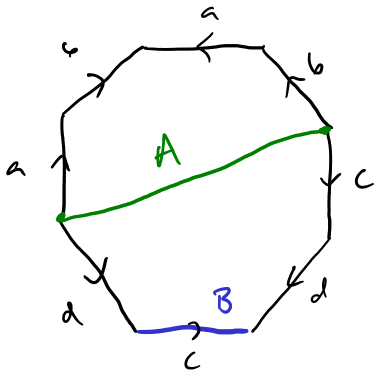
   is an exact sequence, then $A \cong \ker g$ and $D \cong \operatorname{coker} g$.

(Continued on next page.)

2. (Based on §2.1 #17(b), and related to §1.2 #9.)

Let X be the closed orientable surface of genus 2, and let $i\colon A \hookrightarrow X$ and $j\colon B \hookrightarrow X$ be the inclusion of the separating and non-separating circles indicated below:





(a) Draw pictures of X/A and of X/B.

(They don't need to be beautiful pictures, but they should be morally correct.)

(b) Describe $i_*\colon H_1(A) \to H_1(X)$ and $j_*\colon H_1(B) \to H_1(X)$, and find their kernels and cokernels.

Hint: In Lecture 27, I argued that the boundary of the octagon above gives an inclusion $S^1 \vee S^1 \vee S^1 \vee S^1 \hookrightarrow X$ that induces an isomorphism on H_1, and in particular $H_1(X) \cong \mathbb{Z}^4$.

(c) Use the long exact sequence of the pair (X, B) to compute $\tilde{H}_*(X/B)$.

(You may use reduced or unreduced homology of X and B, whichever you prefer.)

(d) Do the same with X/A.

Hint: You may use the fact that any short exact sequence of the form

$$0 \to \mathbb{Z} \to ? \to \mathbb{Z} \to 0$$

is necessarily split, as Hatcher discusses on pages 147–148.