Solutions to Homework 1

1. (a) Let $X$ and $Y$ be topological spaces. Define what it means for a map $f: X \to Y$ to be a quotient map.

Solution: $f$ is surjective, and for every subset $U \subset Y$, $U$ is open if and only if $f^{-1}(U)$ is open.

(b) Let $f: X \to Y$ be a quotient map. Show that for any other space $Z$, a continuous map $g: X \to Z$ descends to $Y$ – that is, there is a continuous map $h: Y \to Z$ with $g = h \circ f$ – if and only if $g$ is constant on the fibers of $f$.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{h} \\
Z & & Z
\end{array}
\]

Solution: First suppose that $g = h \circ f$. If $x, x' \in X$ lie in the same fiber of $f$, then $g(x) = h(f(x)) = h(f(x')) = g(x')$.

Conversely, suppose that $g$ is constant on the fibers of $f$. Given a point $y \in Y$, choose a point $x \in X$ with $f(x) = y$, which is possible because $f$ is surjective, and define $h(y) = g(x)$. Then $h$ is well-defined: if $x' \in X$ is another point with $f(x') = y$, then $g(x') = g(x)$ by hypothesis.

It remains to show that $h$ is continuous. Let $U \subset Z$ be open. Then $f^{-1}(h^{-1}(U)) = g^{-1}(U)$ is open because $g$ is continuous, so $h^{-1}(U)$ is open because $f$ is a quotient map.

(c) Suppose that $X$ is compact, $Y$ is Hausdorff, and $f: X \to Y$ is a continuous surjection. Show that $f$ is a quotient map.

Solution: We will show that $F \subset Y$ is closed if and only if $f^{-1}(F)$ is closed, which is equivalent to the definition because $f^{-1}$ preserves complements. If $F \subset Y$ is closed then $f^{-1}(F)$ is closed because $f$ is continuous. Conversely, suppose that $f^{-1}(F)$ is
closed. A closed subset of a compact space is compact, so $f^{-1}(F)$ is compact. Because $f$ is surjective, we have $F = f(f^{-1}(F))$, and the continuous image of a compact space is compact, so $F$ is compact. A compact subspace of a Hausdorff space is closed, so $F$ is closed.

2. I claim that $(S^1 \times I)/(S^1 \times \{0\})$ is homeomorphic to $D^2$.

(a) Draw a picture.

Solution:

(b) Argue that it is enough to write down a continuous surjection $S^1 \times I \to D^2$ which is constant on $S^1 \times \{0\}$ and injective otherwise.

Solution: Let $f: S^1 \times I \to D^2$ be such a map, and let

$$g: S^1 \times I \to (S^1 \times I)/(S^1 \times \{0\})$$

be the natural map. The fibers of $f$ and $g$ are the same: $S^1 \times \{0\}$ and the single points of $S^1 \times (0, 1]$; thus $f$ is constant on the fibers of $g$ and vice versa. By problem 1(c), $f$ is a quotient map, so by problem 1(b) there is a continuous map

$$h: D^2 \to (S^1 \times I)/(S^1 \times \{0\})$$

with $g = h \circ f$. And $g$ is a quotient map by definition, so there is a continuous map

$$k: (S^1 \times I)/(S^1 \times \{0\}) \to D^2$$

with $f = k \circ g$. Now we have $f = k \circ h \circ f$ and $g = h \circ k \circ g$, and because $f$ and $g$ are surjective we get $k \circ h = 1$ and $h \circ k = 1$. So $h$ and $k$ are inverse homeomorphisms.
(c) Write down such a map.

**Solution:** We regard $S^1$ and $D^2$ as subsets of $\mathbb{R}^2$ in the usual way. The map we want is

\[ S^1 \times I \to D^2, \quad (x, t) \mapsto tx. \]

3. (a) Show that any group homomorphism $\varphi: \mathbb{Z} \to \mathbb{Z}$ is of the form $\varphi(x) = nx$ for some $n \in \mathbb{Z}$.

**Solution:** Let $n = \varphi(1)$. For any $x \in \mathbb{Z}$, write

\[ x \text{ times } x = 1 + \cdots + 1. \]

Then

\[ \varphi(x) = n + \cdots + n = nx. \]

(b) Let $\varphi: \mathbb{Z} \to \mathbb{Z}$ be multiplication by 2. Show that there is no homomorphism $\psi: \mathbb{Z} \to \mathbb{Z}$ such that $\psi \circ \varphi = 1$.

**Solution:** If there were such a $\psi$ then $\varphi$ would be surjective: for every $x \in \mathbb{Z}$ we would have $x = \varphi(\psi(x))$. But $\varphi$ is not surjective: for example, 3 is not in the image of $\varphi$.

(c) Let $G$ be a finite group. Show that any homomorphism $\varphi: G \to \mathbb{Z}$ is trivial.

**Solution:** Let $g \in G$. Because $G$ is finite, there is a positive integer $n$ such that $g^n = 1$. Then

\[ 0 = \varphi(g^n) = n\varphi(g), \]

so $\varphi(g) = 0$. 