Ch. 0 #1. Let $X$ be the solid square $[-1,1] \times [-1,1] \subset \mathbb{R}^2$. First we describe a deformation retraction of $X \setminus 0$ onto its boundary square. Define a map $r: X \setminus 0 \to X \setminus 0$ by projecting from origin: explicitly,

$$r(x, y) = \begin{cases} 
(1, \frac{y}{x}) & \text{if } -x \leq y \leq x \text{ (right quarter)} \\
(x/y, 1) & \text{if } -y \leq x \leq y \text{ (top quarter)} \\
(-1, -y/x) & \text{if } x \leq y \leq -x \text{ (left quarter)} \\
(-x/y, -1) & \text{if } y \leq x \leq -y \text{ (bottom quarter)}.
\end{cases}$$

We see that the denominators are non-zero away from the origin, and the four rules agree on the overlaps, hence they patch together give a continuous function on $X \setminus 0$. Moreover we see that $r$ fixes the boundary square, so the straight-line homotopy $F$ from the identity to $r$ is a deformation retraction.

Next show that $F$ descends to a deformation retraction of the punctured torus $S^1 \times S^1 \setminus (-1, -1)$ onto the wedge of circles $S^1 \vee S^1 = (S^1 \times 1) \cup (1 \times S^1)$. The map

$$p: X \to S^1 \times S^1 \subset \mathbb{C}^2$$

given by $p(x, y) = (e^{i\pi x}, e^{i\pi y})$ is a quotient map, because the domain is compact and the codomain is Hausdorff. It restricts to give quotient maps from $X \setminus 0$ onto $S^1 \times S^1 \setminus (-1, -1)$ and from $\partial X$ onto $S^1 \vee S^1$. The deformation retraction $F$ descends to give the desired deformation retraction $\overline{F}$ as shown:

$$\begin{array}{c}
(X \setminus 0) \times I \\ p \times 1 \end{array} \xrightarrow{F} \xrightarrow{p} \begin{array}{c}
(X \setminus 0) \\ p \end{array} \xrightarrow{F} \begin{array}{c}
(S^1 \times S^1 \setminus (-1, -1)) \times I \\ \overline{F} \end{array} \xrightarrow{p} \begin{array}{c}
S^1 \times S^1 \setminus (-1, -1).
\end{array}$$

The universal property of the quotient topology applies because $p \circ F$ is constant on the fibers of $p \times 1$. 
Ch. 0 #12. First we argue that any continuous map $f: X \to Y$ induces a map $f_*$ from the set of path components of $X$ to the set of path components of $Y$, and similarly with connected components. For a point $x \in X$, we let $[x]$ denote the path component or connected component of $x$, and we define $f_*([x]) = [f(x)]$. This is well-defined, as follows. If $x'$ is another point in the same path component as $x$, let $\gamma: I \to X$ be a path from $x$ to $x'$; then $f \circ \gamma$ is a path from $f(x)$ to $f(x')$, so these lie in the same path component of $Y$. If $x'$ is another point in the same connected component as $x$, let $A \subset X$ be a connected subspace that contains both $x$ and $x'$; then $f(A)$ is a connected subspace of $Y$ that contains both $f(x)$ and $f(x')$, so these lie in the same connected component of $Y$.

Next we argue that if $f \simeq f'$ then $f_* = f'*$ in both cases. Let $F: X \times I \to Y$ be a homotopy from $f$ to $f'$. For path components, we see that for all $x \in X$, the path $F(x, -)$ is a path from $f(x)$ to $f'(x)$, so

$$f_*([x]) = [f(x)] = [f'(x)] = f'_*([x]).$$

For connected components, we see that $F(x \times I)$ is a connected subspace of $Y$ that contains both $f(x)$ and $f'(x)$, so the same equation holds.

Next we observe that if $g: Y \to Z$ is another continuous map, then $g_* \circ f_* = (g \circ f)_*$ for either path components or connected components: for any $x \in X$ we have

$$g_*(f_*([x])) = g_*([f(x)]) = [g(f(x))] = (g \circ f)_*([x]).$$

And the identity map $1: X \to X$ induces the identity map in both cases:

$$1_*([x]) = [1(x)] = [x].$$

Having set all this up, suppose that $f: X \to Y$ is a homotopy equivalence, and let $g: Y \to X$ be a homotopy inverse, so $g \circ f \simeq 1$ and $f \circ g \simeq 1$. Then $g_* \circ f_* = (g \circ f)_* = 1_* = 1$, and similarly $f_* \circ g_* = 1$, so $f_*$ and $g_*$ are inverse bijections.

For the claim about when path components and connected components coincide, observe that the diagram

\[
\begin{array}{ccc}
\text{path components of } X & \xrightarrow{f_*} & \text{path components of } Y \\
\downarrow & & \downarrow \\
\text{connected components of } X & \xrightarrow{f_*} & \text{connected components of } Y
\end{array}
\]

commutes, so if the top, bottom, and left side are bijections, then the right side is a bijection as well.
Regard $S^n$ as embedded in $\mathbb{R}^{n+1}$ in the usual way, and regard $\mathbb{R}^{m+1} \times \mathbb{R}^{n+1} = \mathbb{R}^{m+n+2}$. Consider the map

$$ f : S^m \times S^n \times I \to S^{m+n+1} $$

given by

$$ f(x, y, t) = (\cos(\frac{\pi}{2} t) \cdot x, \sin(\frac{\pi}{2} t) \cdot y). $$

By definition, the join is the quotient of $S^m \times S^n \times I$ that collapses $S^m \times y \times 0$ for each $y \in S^n$, and $x \times S^n \times 1$ for each $x \in S^m$. We will see that $f$ is surjective, and the fibers are exactly these equivalence classes. Because it is a continuous map from a compact space onto a Hausdorff space, it identifies the quotient space of $S^m \times S^n \times I$ with $S^{m+n+1}$.

So we analyze the fibers of $f$. Let $(v, w) \in S^{m+n+1}$, so $v \in \mathbb{R}^{m+1}$, $w \in \mathbb{R}^{n+1}$, and $|v|^2 + |w|^2 = 1$. If $v \neq 0$ and $w \neq 0$, let $\theta$ be the unique angle between 0 and $\frac{\pi}{2}$ with $\cos \theta = |v|$ and $\sin \theta = |w|$; then

$$ f \left( \frac{v}{|v|}, \frac{w}{|w|}, \frac{\theta}{\pi/2} \right) = (v, w), $$

so $f^{-1}(v, w)$ is a single point. If $v = 0$ then $w \in S^n$, and we see that $f^{-1}(0, w)$ is $S^m \times w \times 1$. If $w = 0$ then $v \in S^m$, and we see that $f^{-1}(v, 0)$ is $v \times S^n \times 0$.

$$ S^1 \times S^0 = S^2 $$

§1.1 #5 We can deduce \((a) \iff (b)\) from problem 2 from last week. Consider the diagram

\[
\begin{array}{ccc}
S^1 \times I & \xrightarrow{f} & D^2 \\
\downarrow{F} & & \downarrow{h} \\
\phantom{S^1 \times I} & \xrightarrow{X} & \\
\end{array}
\]

where \(f(z,t) = tz\) collapses \(S^1 \times 0\) to a point. Let \(i: S^1 \hookrightarrow D^1\) be the inclusion of the boundary circle, and let

\[
j: S^1 \times 1 \hookrightarrow S^1 \times I;
\]

Then \(f \circ j = i\). Suppose we are given a map \(g: S^1 \to X\). By definition, \(g\) is nullhomotopic if and only if there is a map \(F: S^1 \times I \to X\) such that \(F(x,0)\) is constant and \(F \circ j = g\). By the universal property of the quotient map \(f\), this happens if and only if there is a map \(h: D^2 \to X\) such that \(h \circ f = F\), and \(h \circ f \circ j = g\). But \(f \circ j = i\), so by definition \(h\) is a map that extends \(g\) in the sense of \((b)\).

We can deduce \((a) \iff (c)\) from problem 6 below. For a point \(x_0 \in X\), let \(P \subset X\) be the path component of \(x_0\), so \(\pi_1(X,x_0) = \pi_1(P,x_0)\). Problem 6 implies that \((c)\) holds if and only if \([S^1,P]\) is a one-point set for every path component \(P \subset X\); note that a group is trivial if and only if all its elements are conjugate. And \([S^1,P]\) is a one-point set if and only if \((a)\) holds; note that any two constant maps \(S^1 \to P\) are homotopic because \(P\) is path connected.

Finally we argue that \(X\) is simply-connected if and only if all maps \(S^1 \to X\) are homotopic. By definition, simply-connected means “path-connected and \((c)\),” which is equivalent to “path-connected and \((a)\).” We see that \(X\) is path-connected if and only if all constant maps to \(X\) are homotopic, so the latter is equivalent to “all maps \(S^1 \to X\) are homotopic.”
§1.1 #6  We will pass back and forth freely between maps $\ell: S^1 \to X$ and maps $\gamma: I \to X$ with $\gamma(0) = \gamma(1)$, as on Worksheet 5: the correspondence is $\gamma = \ell \circ p$, where $p: I \to S^1$ is the quotient map given by $p(s) = e^{2\pi is}$. Similarly we pass between homotopies $\ell_0 \simeq \ell_1$ and homotopies $\gamma_0 \simeq \gamma_1$ that satisfy $F(0,t) = F(1,t)$ for all $t \in I$, using the quotient map $(p \times 1): I \times I \to S^1 \times I$.

First we show that if $X$ is path-connected then $\Phi$ is onto. Suppose that $\gamma: I \to X$ corresponds to a path $\gamma: I \to X$ with $\gamma(0) = \gamma(1)$. Choose a path $p: I \to X$ with $p(0) = x_0$ and $p(1) = \gamma(0)$. I claim that $\Phi$ sends $[p \cdot \gamma \cdot \bar{p}] \in \pi_1(X, x_0)$ to $[\gamma] \in [S^1, X]$, where $\bar{p}$ is the inverse path. To see this, observe that the paths $p \cdot \gamma \cdot \bar{p}$ and $\gamma \cdot p \cdot \bar{p}$ give homotopic maps $S^1 \to X$: just rotate the circle by $120^\circ$. Now $\gamma \cdot p \cdot \bar{p}$ is homotopic rel. endpoints to $\gamma$, as we saw in the construction of $\pi_1$.

Next we show that if $f, g: I \to X$ are loops based at $x_0$ and $\Phi([f]) = \Phi([g])$, then $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Let $F: I \times I \to X$ be a homotopy between $f$ and $g$ satisfying $F(0,t) = F(1,t)$ for all $t \in I$. Let $h: I \to X$ be the path $h(t) = F(0,t)$. Then $h(0) = h(1) = x_0$. I claim that $F$ gives a homotopy rel. endpoints from $f$ to $h \cdot g \cdot \tilde{h}$, so $[f] = [h][g][\tilde{h}] \in \pi_1(X, x_0)$. To prove the claim, pre-compose $F$ with the map $I \times I \to I \times I$ indicated below:

More precisely but less clearly, this is

$$(s, t) \mapsto \begin{cases} 
(0, 3s) & \text{if } t \geq 3s, \\
(1, 3 - 3s) & \text{if } t \geq 3 - 3s, \\
(s - \frac{t}{3}, t) & \text{otherwise.}
\end{cases}$$

*In lecture I called this $p^{-1}$, but here I decided to match Hatcher’s notation.*