Solutions to Homework 3

1. Hatcher, Chapter 0 #11. *Show that* $f\colon X \to Y$ *is a homotopy equivalence if there exist maps* $g, h\colon Y \to X$ *such that* $fg \simeq 1$ *and* $hf \simeq 1$. *More generally, show that* f *is a homotopy equivalence if* fg *and* hf *are homotopy equivalences.*

   Solution: If $hf \simeq 1$ and $fg \simeq 1$, then $g \simeq hfg \simeq h$. Thus $gf \simeq hf \simeq 1$, and we already have $fg \simeq 1$, so g is a homotopy inverse to f.

   More generally, if fg and hf are homotopy equivlances, let $k\colon Y \to Y$ be a homotopy inverse to fg, and let $l\colon X \to X$ be a homotopy inverse to hf. Then $f(gk) = (fg)k \simeq 1$, and $(lh)f = l(hf) \simeq 1$, so f has right and left homotopy inverses, so f a homotopy equivalence by the previous paragraph.

2. Let (X, x) be a pointed space. Show that the reduced suspension ΣX is homeomorphic to the smash product $X \wedge S^1$.

   Solution: The reduced suspension ΣX is obtained by collapsing

   $$(X \times \{0\}) \cup (\{x_0\} \times I) \cup (X \times \{1\}) \subset X \times I.$$

   The smash product $X \wedge S^1$ is obtained by collapsing

   $$(\{x_0\} \times S^1) \cup (X \times \{1\}) \subset X \times S^1.$$

   Consider the quotient map $f\colon I \to S^1$ given by $f(t) = e^{2\pi i t}$, and the diagram

   $$\begin{tikzcd} X \times I \arrow[r, "1\times f"] \arrow[d, "p"'] & X \times S^1 \arrow[d, "q"] \\ \Sigma X & X \wedge S^1 \end{tikzcd}$$

   Then the first set above is $(1 \times f)^{-1}$ of the second set, so $q \circ (1 \times f)$ is constant on the fibers of p and vice versa. Thus $q \circ (1 \times f)$ descends to give a continuous bijection $\Sigma X \to X \wedge S^1$, and if $1 \times f$ is a quotient map then $q \circ (1 \times f)$ is be a quotient map, and we can say that p descends to give a continuous inverse.

   There is a subtle issue here, although I won't mind too much if you gloss over it: knowing that f is a quotient map does not in general imply that $1 \times f$ is a quotient map, but it's true in this case because I is compact and S^1 is Hausdorff. (The relevant slogan is "a proper map is universally closed.") Factor $1 \times f$ as

   $$\begin{array}{ccccc} X \times I & \hookrightarrow & X \times I \times S^1 & \twoheadrightarrow & X \times S^1 \\ (x,t) & \mapsto & (x,t,f(t)) & & \\ & & (x,t,z) & \mapsto & (x,z). \end{array}$$

   The first map is the embedding of a closed subspace: its image is X times the graph of f, and the graph of f is closed in $I \times S^1$ because S^1 is Hausdorff. Hence the first map is a closed map. The second is also a closed map, because it is a projection from a product, and the factor I that's being forgotten is compact. So the composition $1 \times f$ is a closed map, and a closed continuous surjection is a quotient map.

3. Show that if $n \neq 0$, then the map $f: S^1 \to S^1$ given by $f(z) = z^n$ is not nullhomotopic.

Solution: Suppose that $F: S^1 \times I \to S^1$ is a (free) homotopy from f to a constant map. Then the map $G: S^1 \times I \to S^1$ given by

$$G(z,t) = \frac{F(z,t)}{F(1,t)}$$

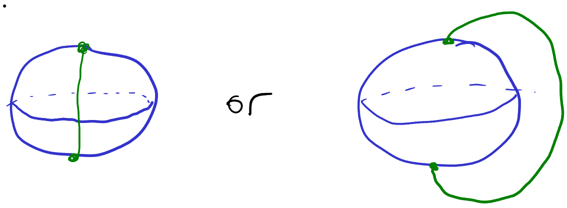
is a homotopy rel. basepoint from f to the constant map at 1: we see that $G(1,t) = 1$ for all t. From Worksheet 5 #1 we know that f is homtopic rel. basepoint to a constant map if and only if the path $\gamma: I \to S^1$ given by $\gamma(s) = e^{2\pi i n s}$ is homotopic rel. endpoints to a constant map, and in class we proved that if $n \neq 0$ then it's not.

Alternatively, we could hit it with a big hammer and appeal to §1.1 #6 from last week's homework. That problem said that the map from $\pi_1(S^1, 1)$ to free homotopy classes of maps $[S^1, S^1]$ is surjective, and the fibers are conjugacy classes. But $\pi_1(S^1, 1) \cong \mathbb{Z}$ is Abelian, so conjugacy classes are just single elements, so the map is a bijection. Now f represents n in $\pi_1(S^1, 1)$, and a constant path at 1 represents 0, so the two are freely homotopic if and only if $n = 0$.

4. Draw pictures of each of the following spaces:

   (a) The sphere S^2 union the line segment connecting the north and south poles.

   Solution:

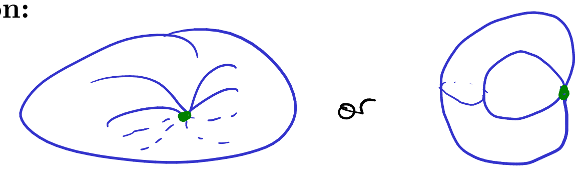


   (b) The wedge sum $S^2 \vee S^1$.

   Solution:

   (c) The sphere S^2 with the north and south poles glued together.

   Solution:

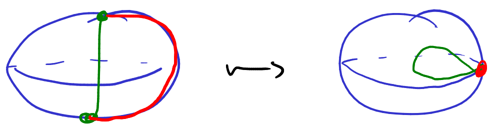


   (d) The torus $S^1 \times S^1$ with a disc D^2 glued in along the circle $S^1 \times \{1\}$.

   Solution:

Argue (loosely) that all four are homotopy equivalent, because you can get between them by collapsing some nice contractible subspaces. Later we'll try to pin this down more rigorously.

Solution: To get from (a) to (b), collapse a meridian:



To get from (a) to (c), collapse the line segment. To get from (d) to (c), collapse the disc.