§2.1 #12. Show that chain homotopy of chain maps is an equivalence relation.

Let $A_\bullet$ and $B_\bullet$ be two chain complexes, and let $f_\bullet$ be a chain map:

$$\cdots \longrightarrow A_{i+1} \xrightarrow{\partial_{i+1}} A_i \xrightarrow{\partial_i} A_{i-1} \longrightarrow \cdots$$

$$\cdots \longrightarrow B_{i+1} \xrightarrow{\partial_{i+1}} B_i \xrightarrow{\partial_i} B_{i-1} \longrightarrow \cdots$$

**Reflexive:** The zero map is a chain homotopy from $f_\bullet$ to $f_\bullet$: that is, if we take $P_i = 0$: $A_i \rightarrow B_{i+1}$, then $f_i - f_i = d_{i+1} \circ P_i + P_{i-1} \circ d_i$, so $f_\bullet \simeq f_\bullet$.

**Symmetric:** If $P_\bullet$ is a chain homotopy from $f_\bullet$ to another chain map $g_\bullet$, then $-P_\bullet$ is a chain homotopy from $g_\bullet$ to $f_\bullet$: that is, given maps $P_i: A_i \rightarrow B_{i+1}$ with $f_i - g_i = d_{i+1} \circ P_i + P_{i-1} \circ d_i$, we see that

$$g_i - f_i = d_{i+1} \circ (-P_i) + (-P_{i-1}) \circ d_i.$$

**Transitive:** If $Q_\bullet$ is a chain homotopy from $g_\bullet$ to a third chain map $h_\bullet$, then $P_\bullet + Q_\bullet$ is a chain homotopy from $f_\bullet$ to $h_\bullet$: that is, given maps $Q_i: A_i \rightarrow B_{i+1}$ with $g_i - h_i = d_{i+1} \circ Q_i + Q_{i-1} \circ d_i$, we see that

$$f_i - h_i = (f_i - g_i) + (g_i - h_i) = d_{i+1} \circ (P_i + Q_i) + (P_{i-1} + Q_{i-1}) \circ d_i.$$

1
§2.1 #15. For an exact sequence $A \to B \to C \to D \to E$ show that $C = 0$ iff the map $A \to B$ is surjective and $D \to E$ is injective.

Label the maps as

$$A \overset{f}{\to} B \overset{g}{\to} C \overset{h}{\to} D \overset{k}{\to} E.$$ 

First suppose that $C = 0$. Then $g = 0$, so $\ker g = B$, so $\text{im} f = B$ because the sequence is exact at $B$, so $f$ is surjective. And $h = 0$, so $\text{im} h = 0$, so $\ker k = 0$ because the sequence is exact at $D$, so $k$ is injective.

Conversely, suppose that $f$ is surjective and $k$ is injective. Then $\text{im} f = B$, so $\ker g = B$ because the sequence is exact at $B$, so $g = 0$, so $\text{im} g = 0$. And $k = 0$, so $\ker h = 0$ because the sequence is exact at $D$, so $h = 0$, so $\ker h = C$. But the sequence is exact at $C$, so $\ker h = \text{im} g$, so $C = 0$.

Hence for a pair of spaces $(X, A)$, the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups iff $H_n(X, A) = 0$ for all $n$.

The long exact sequence of the pair includes

$$H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(X).$$

The maps $H_n(A) \to H_n(X)$ are isomorphisms for all $n$ if and only if they are both injective and surjective for all $n$. By reindexing, this is true if and only if the leftmost map in our five-term exact sequence is surjective and the rightmost map is injective for all $n$. But by the first part of the problem, this is true if and only if the middle group vanishes for all $n$. 

2
Show that $\tilde{H}_n(X) \approx \tilde{H}_{n+1}(SX)$ for all $n$, where $SX$ is the suspension of $X$.

In Chapter 0, Hatcher defined the cone $CX$ as $(X \times I)/(X \times \{0\})$. It is contractible: we see that $X \times I$ deformation retracts onto $X \times \{0\}$ via the homotopy $f_t(x, s) = (x, ts)$, and this descends to a deformation retraction of $CX$ onto the cone point.

He defined the suspension $SX$ as $(X \times I)/(X \times \{0, 1\})$, and we will view it as the quotient of $CX$ by the image of $X \times \{1\}$, which is homeomorphic to $X$ and is a good pair in the sense of Theorem 2.13 or Proposition 2.22: the image of $X \times (0, 1]$ in $CX$ deformation retracts onto the image of $X \times \{1\}$ via a homotopy like the one above.

Thus we get a long exact sequence

$$\cdots \to \tilde{H}_n(CX) \to \tilde{H}_n(SX) \to \tilde{H}_{n-1}(X) \to \tilde{H}_{n-1}(CX) \to \cdots$$

Because $CX$ is contractible, we have $\tilde{H}_n(CX) = 0$ for all $n$, so the connecting homomorphism $\tilde{H}_n(SX) \to \tilde{H}_{n-1}(X)$ is both injective and surjective for all $n$, so it is an isomorphism.

More generally, thinking of $SX$ as the union of two cones $CX$ with their bases identified, compute the reduced homology groups of the union of $n$ cones $CX$ with their bases identified.

There are a couple of ways to approach this, but here is one.

Let $C_1, \ldots, C_n$ be $n$ copies of $CX$, and let $Y$ be the space obtained by gluing them all together along the images of $X \times \{1\}$ in $CX$. Then we see that

$$\frac{Y}{C_1} \cong \frac{SX \vee SX \vee \cdots \vee SX}{\},$$

and the long exact sequence of the pair $(Y, C_1)$ includes

$$\tilde{H}_n(C_1) \to H_n(Y) \to \tilde{H}_n(Y/C_1) \to \tilde{H}_{n-1}(C_1),$$

so

$$\tilde{H}_n(Y) \cong \tilde{H}_n(Y/C_1)$$

$$\cong \tilde{H}_n(SX \vee SX \vee \cdots \vee SX)$$

$$\cong \tilde{H}_n(SX) \oplus \tilde{H}_n(SX) \oplus \cdots \oplus \tilde{H}_n(SX)$$

$$\cong \tilde{H}_{n-1}(X) \oplus \tilde{H}_{n-1}(X) \oplus \cdots \oplus \tilde{H}_{n-1}(X),$$

where the direct sum is taken $n-1$ times.
**Challenge:** §2.1 #21. Making the preceding problem more concrete, construct explicit chain maps \( s : C_n(X) \to C_{n+1}(SX) \) inducing isomorphisms \( \hat{H}_n(X) \to \hat{H}_{n+1}(SX) \).

The idea is illustrated in the picture below: given a triangle in \( X \), we get a tetrahedron above it in \( SX \), and another tetrahedron below it, and we want to take their difference.

Given an \( n \)-simplex \( \sigma : \Delta^n \to X \), we can take the suspension \( S\sigma : S\Delta^n \to SX \). Divide \( S\Delta^n \) into two halves, the images of \( \Delta^n \times [0, \frac{1}{2}] \) and \( \Delta^n \times [\frac{1}{2}, 1] \), and choose homeomorphisms \( f \) from \( \Delta^{n+1} \) to the first half and \( g \) from \( \Delta^{n+1} \) to the second half. If we choose \( f \) and \( g \) well, then the map \( C_n(X) \to C_{n+1}(X) \) determined by

\[
\sigma \mapsto S\sigma \circ f - S\sigma \circ g
\]

is a chain map, and induces an isomorphism on homology. But it’s late and this problem wasn’t required, so I’m just going to assert that the details “can be checked.”

![Diagram](image.png)

Here’s another possibility, although maybe it’s not quite in the spirit of the problem since it uses the big hammer of Proposition 2.22. Identify \( \hat{H}_{n+1}(SX) \) with \( H_{n+1}(CX, X) \), where \( X \) is still embedded in \( CX \) as the image of \( X \times \{1\} \). Given an \( n \)-simplex \( \sigma : \Delta^n \to X \), we can take the cone \( C\sigma : C\Delta^n \to CX \). Choose a homeomorphism \( h : \Delta^{n+1} \to C\Delta^n \); then the map \( C_n(X) \to C_{n+1}(CX) \) determined by

\[
\sigma \mapsto C\sigma \circ h
\]

is not a chain map, but if instead we map \( C_n(X) \to C_{n+1}(CX, X) \) then it is a chain map. And the induced map \( H_n(X) \to H_{n+1}(CX, X) \) turns out to be inverse to the connecting homomorphism \( H_{n+1}(CX, X) \to H_n(X) \), hence is an isomorphism.