Midterm 1

Due Friday, October 30, 2020

You may refer to Hatcher's book, the lecture notes, and the homework problems and solutions, and you may quote any results proved there. But do not consult your colleagues or the internet.

1. (a) Give an example of path-connected spaces X and Y and a map $f\colon X \to Y$ which is injective but not surjective, for which the induced map $f_*\colon \pi_1(X, x) \to \pi_1(Y, f(x))$ is injective but not surjective for some (and hence every) $x \in X$.

   (b) Give an example where f is injective but not surjective and f_* is surjective but not injective.

   (c) Give an example where f is surjective but not injective and f_* is injective but not surjective.

   (d) Give an example where f is surjective but not injective and f_* is surjective but not injective.

   (e) Give one more example along these lines that you think is interesting.

2. For a map $f\colon X \to Y$, the *mapping cylinder* M_f is defined on page 2 of Hatcher's book. I claim that if f is homotopic to another map $g\colon X \to Y$, then M_f and M_g are homotopy equivalent rel. $X \times \{0\}$.

   (a) Write down explicit maps $\Phi\colon M_g \to M_f$ and $\Psi\colon M_f \to M_g$ that will turn out to be homotopy inverse to one another rel. $X \times \{0\}$. Be clear about why your maps are well-defined, but don't belabor the point.

   (b) Write out the composition $\Phi \circ \Psi$ explicitly. Assert that it is homotopic to the identity, but don't write an explicit homotopy.

   (c) If X is a point, then a homotopy from f to g is the same as a path from $f(x)$ to $g(x)$. Draw a picture of what Φ, Ψ, and $\Psi \circ \Phi$ look like in that case, to make it plausible that $\Psi \circ \Phi$ is homotopic to the identity. (You may want to do this part first.)