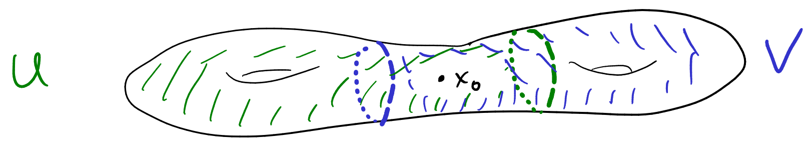
Solutions to Midterm 2

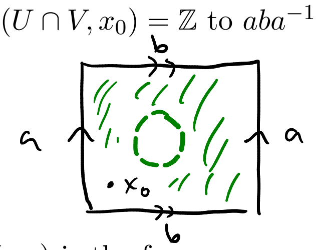
1. Let X be the orientable surface of genus 2, and let x_0 be a point in the middle of the "neck."

   (a) Compute $\pi_1(X, x_0)$ by cutting X into two punctured tori and applying van Kampen's theorem.

   Solution: Let U be a neighborhood of the left half, and let V be a neighborhood of the right half, so $U \cap V$ contains x_0 and is homeomorphic to $S^1 \times (-\epsilon, \epsilon)$. All three are path-connected, and $X = U \cup V$.



   Then U is homeomorphic to a torus minus a closed disc, so from Chapter 0 #1 we know that U deformation retracts onto $S_1 \vee S_1$, so $\pi_1(U, x_0)$ is the free group on two letters, say a and b. Moreover, if k is the inclusion $U \cap V \hookrightarrow U$, then k_* takes a generator of $\pi_1(U \cap V, x_0) = \mathbb{Z}$ to $aba^{-1}b^{-1}$:



   Similarly, $\pi_1(V, x_0)$ is the free group on two letters, say c and d, and if $l \colon U \cap V \hookrightarrow V$ then l_* takes a generator of $\pi_1(U \cap V, x_0)$ to $cdc^{-1}d^{-1}$.

   Now van Kampen's theorem says that

$$\pi_1(X, x_0) = \langle a, b, c, d \mid aba^{-1}b^{-1} = cdc^{-1}d^{-1} \rangle.$$

(b) Show that X is not homeomorphic to S^2 or $S^1 \times S^1 \ldots$

Solution: Let F be the free group on two letters e and f, and let

$$\varphi\colon \pi_1(X, x_0) \to F$$

be the homomorphism determined by

$$\begin{aligned} a &\mapsto e \\ b &\mapsto 1 \\ c &\mapsto f \\ d &\mapsto 1. \end{aligned}$$

This is well-defined, because

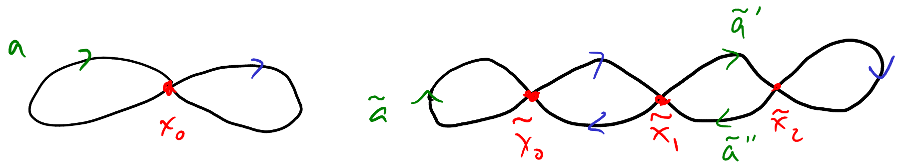
$$\varphi(aba^{-1}) = e \cdot e^{-1} = 1 = f \cdot f^{-1} = \varphi(cdc^{-1}),$$

and it is surjective, because the image includes both generators, so F is a quotient of $\pi_1(X, x_0)$. A quotient of an Abelian group is Abelian, and F is not Abelian, so neither is $\pi_1(X, x_0)$. But $\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$ and $\pi_1(S^2) = 0$ are Abelian, so X is not even homotopy equivalent to the torus or the sphere, much less homeomorphic.

2. Let $X = S^1 \vee S^1$, let $x_0 \in X$ be the wedge point, let $p\colon \tilde{X} \to X$ be the three-sheeted cover shown below, and let $p^{-1}(x_0) = \{\tilde{x}_0, \tilde{x}_1, \tilde{x}_2\}$ as shown. Let $\varphi\colon \tilde{X} \to \tilde{X}$ be a deck transformation, that is, a homeomorphism with $p \circ \varphi = p$.

   (a) Show that we cannot have $\varphi(\tilde{x}_0) = \tilde{x}_1$ or $\varphi(\tilde{x}_0) = \tilde{x}_2$.

   Solution: Let $a\colon I \to X$ be the path from x_0 to x_0 as labeled in the picture, and let $\tilde{a}, \tilde{a}', \tilde{a}''\colon I \to X$ be the following paths in $\tilde{X}$:



   The three paths are lifts of a: that is, $p \circ \tilde{a} = p \circ \tilde{a}' = p \circ \tilde{a}'' = a$. If $\varphi\colon \tilde{X} \to \tilde{X}$ is a deck transformation, we see that the paths $\varphi \circ \tilde{a}$, $\varphi \circ \tilde{a}'$, and $\varphi \circ \tilde{a}''$ also lift a.

   If $\varphi(\tilde{x}_0) = \tilde{x}_1$ then $\varphi \circ \tilde{a}$ and $\tilde{a}'$ are both paths that lift a and start from $\tilde{x}_1$, so by unique path lifting (after Proposition 1.30) they must be equal. But $\varphi(\tilde{a}(1)) = \varphi(\tilde{x}_0) = \tilde{x}_1$, whereas $\tilde{a}'(1) = \tilde{x}_2$.

   Similarly, if $\varphi(\tilde{x}_0) = \tilde{x}_2$ then $\varphi \circ \tilde{a}$ and $\tilde{a}''$ are both paths that lift a and start from $\tilde{x}_2$, so by unique path lifting they must be equal. But $\varphi(\tilde{a}(1)) = \varphi(\tilde{x}_0) = \tilde{x}_2$, whereas $\tilde{a}''(1) = \tilde{x}_1$.

   (b) Conclude that φ is the identity.

   Solution: Because $p \circ \varphi = p$, we have $p(\varphi(\tilde{x}_0)) = p(\tilde{x}_0) = x_0$, so $\varphi(\tilde{x}_0) \in p^{-1}(x_0)$, so from part (a) we conclude that $\varphi(\tilde{x}_0) = \tilde{x}_0$.

   Now we apply the unique lifting property (Proposition 1.34), taking $p\colon \tilde{X} \to X$ both as the covering map and the map to be lifted: our deck transformation $\varphi\colon \tilde{X} \to \tilde{X}$ and the identity $1\colon \tilde{X} \to \tilde{X}$ both lift p, and they agree at the point $\tilde{x}_0$, so they are equal.

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (\tilde{X}, \tilde{x}_0) \arrow[ur, shift left, "\varphi"] \arrow[ur, shift right, "1"'] \arrow[r, "p"'] & (X, x_0) \end{tikzcd}$$