Worksheet 10

Math 634, Algebraic Topology

Wednesday, October 21, 2020

In lecture I said that if $n \geq 2$ then every path $\gamma\colon I \to S^n$ is homotopic (rel. endpoints) to one that is not surjective. Work through the following proof, convincing yourselves that each claim is correct. Be skeptical! And be sure to draw enough pictures as you go.

Choose a point $x \in S^n$ that's not equal to $\gamma(0)$ or $\gamma(1)$. Let U be the intersection of S^n with a small open ball around x. Then $\gamma^{-1}(U) \subset I$ is a union of disjoint open intervals. And $\gamma^{-1}(x) \subset I$ is covered by those open intervals, and is compact, so we can extract a finite subcover. Thus we have

$$0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k < 1$$

where $\gamma((a_i, b_i)) \subset U$ for all i, and if $\gamma(s) = x$ then $s \in (a_i, b_i)$ for some i.

Now we have finitely many problems, so we can fix them one at a time. For each i, choose a path $\gamma_i'\colon [a_i, b_i] \to \bar{U}$ that avoids x and satisfies

$$\gamma_i'(a_i) = \gamma(a_i) \qquad\qquad \gamma_i'(b_i) = \gamma(b_i)$$

for all i. Define

$$\gamma'(s) = \begin{cases} \gamma_i'(s) & \text{if } s \in [a_i, b_i] \text{ for some } i, \\ \gamma(s) & \text{otherwise.} \end{cases}$$

Then γ' is well-defined, continuous, and not surjective. Moreover, $\gamma \simeq \gamma'$ rel. endpoints: on the intervals $[0, a_1]$ and $[b_i, a_{i+1}]$ and $[b_k, 1]$ we have $\gamma = \gamma'$; on the intervals $[a_i, b_i]$, both γ and γ' take values in $\bar{U}$ which homeomorphic to a closed disc, so we can do the straight-line homotopy.

(Where did we use the fact that $n \geq 2$?)