Worksheet 11

Math 634, Algebraic Topology

Friday, October 23, 2020

In lecture we stated van Kampen's theorem: Suppose we write X as a union of two open sets $U \cup V$, where U, V, and $U \cap V$ are path-connected. Label the inclusions as shown:

$$\begin{tikzcd} U \cap V \arrow[r, hook, "k"] \arrow[d, hook, "l"'] & U \arrow[d, hook, "i"] \\ V \arrow[r, hook, "j"'] & X \end{tikzcd}$$

Then the map

$$i_* * j_* \colon \pi_1(U) * \pi_1(V) \to \pi_1(X)$$

is surjective, and the kernel is the normal subgroup generated by

$$\left\{ k_*[\gamma] * l_*[\gamma]^{-1} : [\gamma] \in \pi_1(U \cap V) \right\}.$$

1. Use van Kampen's theorem to show that $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$, the free group on two generators.

   Hint: Start by drawing a picture of $S^1 \vee S^1$. Let U consist of one circle and a neighborhood of the wedge point.

2. At the beginning of the hour we claimed that π_1 of the Klein bottle is the finitely presented group

   $$\langle a, b \mid abab^{-1} = 1 \rangle.$$

   Deduce this from van Kampen's theorem: draw the usual picture of the square with arrows on the edges to indicate gluings, let U be the interior of the square, and let V be a neighorhood of the boundary, so V is a thickening of $S^1 \vee S^1$ and $U \cap V$ is a thickening of S^1.

   If you have time, use this to show that the Klein bottle is not homotopy equivalent to the torus.