Worksheet 12

Math 634, Algebraic Topology

Wednesdsay, October 28, 2020

We define projective space $\mathbb{RP}^n$ as the set of 1-dimensional subspaces of $\mathbb{R}^{n+1}$ – or lines through the origin, if you like that description better – with the quotient topology from the surjection

$$\begin{aligned} \mathbb{R}^{n+1} \setminus 0 &\to \mathbb{RP}^n \\ v &\mapsto \text{the subspace } \mathbb{R}v \subset \mathbb{R}^{n+1}. \end{aligned}$$

The point of this worksheet is to get a feeling for $\mathbb{RP}^n$. Convince yourselves that the following claims are correct.

1. The quotient topology from the surjection
$$\begin{aligned} S^n &\to \mathbb{RP}^n \\ v &\mapsto \text{the subspace } \mathbb{R}v \subset \mathbb{R}^{n+1} \end{aligned}$$
is the same. Thus $\mathbb{RP}^n$ is compact. We can describe $\mathbb{RP}^n$ as S^n with antipodal points identified, or (to put it another way) as $S^n/\mathbb{Z}_2$, where the non-trivial element of $\mathbb{Z}_2$ acts freely on S^n by the antipodal map $v \mapsto -v$.

2. Draw a picture of the quotient map $S^0 \to \mathbb{RP}^0$.

3. In $\mathbb{R}^2$, draw the upper half of the unit circle and some lines through the origin. Convince yourself that $\mathbb{RP}^1 \cong S^1$, and the quotient map $S^1 \to \mathbb{RP}^1$ corresponds to the map $S^1 \to S^1$ given by $z \mapsto z^2$.

4. In $\mathbb{R}^3$, draw with the upper half of the unit sphere and some lines through the origin. Convince yourself that $\mathbb{RP}^2$ can be obtained from the disc D^2 by gluing together opposite points on the boundary circle.

5. More generally, $\mathbb{RP}^n$ can be obtained from the disc D^n by gluing together opposite points on the boundary S^{n-1}, or to put it another way, by taking $\mathbb{RP}^{n-1}$ and attaching a disc D^n via the quotient map $S^{n-1} \to \mathbb{RP}^{n-1}$. Thus for $n \geq 3$ we have $\pi_1(\mathbb{RP}^n) = \pi_1(\mathbb{RP}^{n-1})$, which by induction is $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$

6. So living in $\mathbb{RP}^3$ is like living in a big spherical tank of water, but when you try to swim through the wall you come out of the opposite wall upside down, like Pac-Man's crazy uncle.