Worksheet 14

Math 634, Algebraic Topology

Friday, November 6, 2020

We continue to assume that all our spaces are path-connected and locally path-connected.

In lecture, given a covering space $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and a continuous map $f\colon (Y, y_0) \to (X, x_0)$ with $f_*\pi_1(Y, y_0) \subset p_*\pi_1(\tilde{X}, \tilde{x}_0)$, we constructed a map $\tilde{f}\colon (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ with $p \circ \tilde{f} = f$:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Here you will show that $\tilde{f}$ is continuous. Convince yourselves of that each claim is correct. Be sure to draw as many diagrams and pictures as you need!

1. It is enough to show that every $y \in Y$ has a neighborhood V such that $\tilde{f}|_V$ is continuous.
2. Fix some $y \in Y$. Let $U \subset X$ be a neighborhood of $f(y)$ such that $p^{-1}(U) \cong U \times$ a discrete space. Then there is a continuous map $s\colon U \to \tilde{X}$ with $s(f(y)) = \tilde{f}(y)$ and $p \circ s = 1_U$.
3. There is a path-connected open set $V \subset Y$ with $y \in V \subset f^{-1}(U)$.
4. Now for the meat of the argument, I claim that $\tilde{f}|_V = s \circ f|_V$:
   (a) Let γ be a path from y_0 to y, and let $\tilde{\gamma}$ be the lift of $f \circ \gamma$ starting from $\tilde{x}_0$. By construction, $\tilde{f}(y) = \tilde{\gamma}(1)$.
   (b) For any $y' \in V$, choose a path γ' from y to y'. Then $\gamma \cdot \gamma'$ is a path from y_0 to y', and $\tilde{\gamma} \cdot (s \circ f \circ \gamma')$ lifts $f \circ (\gamma \cdot \gamma')$.
   (c) So $\tilde{f}(y') = s(f(y'))$.
5. Finally, $s \circ f|_V$ is continuous, so $\tilde{f}|_V$ is continuous, so we win.