Worksheet 18

Math 634, Algebraic Topology

Friday, November 20, 2020

1. Let $X = S^2$ and $A = \{2 \text{ points}\} \subset X$. Draw a picture of X/A. Use the long exact sequence of the pair to compute $\tilde{H}_*(X/A)$ and then $H_*(X/A)$.

$$
\begin{tikzcd}
0 \arrow[r] & H_2(A) \arrow[r] & H_2(X) \arrow[r] & H_2(X,A) \arrow[dll, out=-10, in=170] \\
& H_1(A) \arrow[r] & H_1(X) \arrow[r] & H_1(X,A) \arrow[dll, out=-10, in=170] \\
& H_0(A) \arrow[r] & H_0(X) \arrow[r] & H_0(X,A) \arrow[r] & 0,
\end{tikzcd}
$$

2. On Wednesday we discussed a diagram

$$
\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_2(A) \arrow[r, "\partial"] \arrow[d, "i_\sharp"] & C_1(A) \arrow[r, "\partial"] \arrow[d, "i_\sharp"] & C_0(A) \arrow[r] \arrow[d, "i_\sharp"] & 0 \\
0 \arrow[r] & C_2(X) \arrow[r, "\partial"] \arrow[d, "q"] & C_1(X) \arrow[r, "\partial"] \arrow[d, "q"] & C_0(X) \arrow[r] \arrow[d, "q"] & 0 \\
0 \arrow[r] & C_2(X,A) \arrow[r, "\partial"] \arrow[d] & C_1(X,A) \arrow[r, "\partial"] \arrow[d] & C_0(X,A) \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}
$$

whose columns are exact. Convince yourselves that the generator of $H_1(X,A) = \mathbb{Z}$ can be represented by a chain $\alpha \in C_1(X)$ with $\partial(q(\alpha)) = 0$, and that this is equivalent to having $\partial\alpha = i_\sharp\beta$ for some $\beta \in C_0(A)$. Draw a picture of α and β.

If you have time, pick another representative $\alpha' \in C_1(X)$ of the same generator, and convince yourselves that $q(\alpha) - q(\alpha')$ is in the image of ∂.