Worksheet 20

Math 634, Algebraic Topology

Wednesday, November 25, 2020

We have seen that for $n \geq 1$,

$$H_i(S^n) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} & i = n \\ 0 & \text{otherwise,} \end{cases}$$

which we proved by induction using the long exact sequence of the pair (D^n, S^{n-1}).

Let $f\colon S^n \to S^n$ be the reflection $f(x_0, x_1, \ldots, x_n) = (-x_0, x_1, \ldots, x_n)$. I claim that $f_*\colon H_n(S^n) \to H_n(S^n)$ is multiplication by -1:

1. Convince yourself that the identification $S^n \cong D^n/S^{n-1}$ can be chosen so that f corresponds to the reflection $g\colon D^n \to D^n$ given by $g(y_1, y_2 \ldots, y_n) = (-y_1, y_2, \ldots, y_n)$.

2. Prove the claim for $n = 1$, using the diagram

$$\begin{tikzcd} H^1(D^1) \arrow[r] \arrow[d, "g_*"'] & \tilde{H}^1(S^1) \arrow[r] \arrow[d, "f_*"'] & H^0(S^0) \arrow[r] \arrow[d, "g_*"'] & H^0(D^1) \arrow[d, "g_*"'] \\ H^1(D^1) \arrow[r] & \tilde{H}^1(S^1) \arrow[r] & H^0(S^0) \arrow[r] & H^0(D^1). \end{tikzcd}$$

3. Prove the claim for $n \geq 2$ by induction, using the diagram

$$\begin{tikzcd} H^n(D^n) \arrow[r] \arrow[d, "g_*"'] & \tilde{H}^n(S^n) \arrow[r] \arrow[d, "f_*"'] & H^{n-1}(S^{n-1}) \arrow[r] \arrow[d, "g_*"'] & H^{n-1}(D^n) \arrow[d, "g_*"'] \\ H^n(D^n) \arrow[r] & \tilde{H}^n(S^n) \arrow[r] & H^{n-1}(S^{n-1}) \arrow[r] & H^{n-1}(D^n). \end{tikzcd}$$

Thus the antipodal map $a\colon S^n \to S^n$ induces the identity on H_n if n is odd, and multiplication by -1 if n is even. In fact the antipodal map is homotopic to the identity if n is odd, but this shows that it is not homotopic to the identity if n is even.