Worksheet 7

Math 634, Algebraic Topology

Wednesday, October 14, 2020

We have been considering the map $p\colon \mathbb{R} \to S^1$ given by $p(x) = e^{2\pi i x}$.

On Monday we showed that for every path $\gamma\colon I \to S^1$ with $\gamma(0) = \gamma(1) = 1$, there is a unique path $\tilde{\gamma}\colon I \to \mathbb{R}$ with $\tilde{\gamma}(0) = 0$ and $p \circ \tilde{\gamma} = \gamma$.

Today we showed that if two such paths γ_0 and γ_1 are homotopic rel. endpoints, then $\tilde{\gamma}_0(1) = \tilde{\gamma}_1(1)$. Thus the map

$$\begin{aligned} \varphi\colon \pi_1(S^1, 1) &\to \mathbb{Z} \\ [\gamma] &\mapsto \tilde{\gamma}(1) \end{aligned}$$

is well-defined. Now you will show that it is an isomorphism of groups.

3. Show that φ is injective: If $\gamma_0, \gamma_1\colon I \to S^1$ are two loops based at 1 with $\tilde{\gamma}_0(1) = \tilde{\gamma}_1(1)$, then $\gamma_0 \simeq \gamma_1$ rel. endpoints.

   (Let $F\colon I \times I \to \mathbb{R}$ be the straight-line homotopy from $\tilde{\gamma}_0$ to $\tilde{\gamma}_1$. Draw a picture. Convince yourselves that $p \circ F$ is a homotopy rel. endpoints from γ_0 to γ_1. Where did you use the fact that $\tilde{\gamma}_0(1) = \tilde{\gamma}_1(1)$?)

4. Show that φ is surjective: For every $n \in \mathbb{Z}$ there is a loop $\gamma\colon I \to S^1$ such that $\tilde{\gamma}(1) = n$.

   (Just write down a suitable $\tilde{\gamma}$ and let $\gamma = p \circ \tilde{\gamma}$. Draw a picture.)

5. Show that φ is a group homomorphism: Let $\gamma_1, \gamma_2\colon I \to S^1$ be loops based at 1, with $\tilde{\gamma}_1(1) = m$ and $\tilde{\gamma}_2(1) = n$; then the unique map $\alpha\colon I \to \mathbb{R}$ that lifts $\gamma_1 \cdot \gamma_2$ satisfies $\alpha(1) = m + n$.

   (Convince yourselves that

   $$\alpha(t) = \begin{cases} \tilde{\gamma}_1(2t) & \text{if } t \in [0, \frac{1}{2}], \\ m + \tilde{\gamma}_2(2t-1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

   does the job. Draw a picture.)