Final Exam

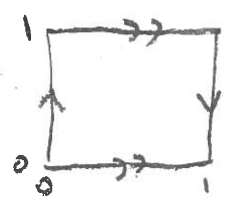
Math 637

Wednesday, December 8, 2021

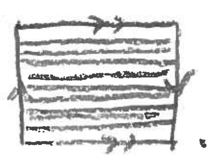
All manifolds, maps, actions, etc. are smooth. You may refer to a printed copy of the book.

1. (a) Let $f\colon M \to N$ be a submersion. Prove that the fibers of f give a foliation of M.

   (b) Let M be the Klein bottle, obtained as a quotient of the square in the usual way,



   and consider the foliation



   Describe the leaf that passes through the central point $(\frac{1}{2}, \frac{1}{2})$, and the leaf that passes through $(\frac{1}{2}, \frac{1}{3})$. (An informal description is fine. It would be good to draw a picture. Notice that the left and right ends of the square have been glued with a half twist.)

   Prove that the leaves of this folation are not the fibers of any submersion $f\colon M \to S^1$. (Hint: I can think of several approaches. You might use the fact that a submersion has local sections, or that a submersion is locally of a certain form, or you could prove by hand that any smooth map $f\colon M \to S^1$ that is constant on the leaves of the foliation must fail to be a submersion at $(\frac{1}{2}, \frac{1}{2})\ldots$)

2. Adapted from Problem 20-13: Characterization of Lie algebra actions that correspond to transitive Lie group actions.

   Suppose we have an action of a finite-dimensional Lie algebra $\mathfrak{g}$ on a smooth manifold M: that is, we have a homomorphism from $\mathfrak{g}$ to the Lie algebra of vector fields on M. Given $X \in \mathfrak{g}$, let $\hat{X}$ be the corresponding vector field on M. Say that the Lie algebra action is *transitive* if for every $p \in M$, the vectors $\hat{X}_p \in T_pM$ span T_pM as $X \in \mathfrak{g}$ varies.

   We have seen that a right action of a Lie group G on a manifold M gives rise to an action of its Lie algebra $\mathfrak{g}$ on M, by taking

$$\hat{X}_p = D(\theta^{(p)})_1(X),$$

   where $\theta^{(p)}: G \to M$ is the orbit map $g \mapsto p \cdot g$. In lecture this was our definition, and in Lee's book it's equation (20.8) on page 526.

   Argue that the Lie algebra action is transitive if and only if the orbit maps $\theta^{(p)}$ are submersions. Prove that if M is connected, then the Lie group action is transitive if and only if the Lie algebra action is transitive in the sense above.

   (Hints: On the midterm you proved that an equivariant map where the group acts transitively on the domain must have constant rank. And at one point you'll want to remember that a submersion is an open map.)