Solutions to Final Exam

Math 637

Wednesday, December 8, 2021

1. (a) Let $f\colon M \to N$ be a submersion. Prove that the fibers of f give a foliation of M.

   Solution: Let $m = \dim M$ and $n = \dim N$, so $m \geq n$. By the Rank Theorem (Theorem 4.12), for every $p \in M$ there are coordinate charts near p and $f(p)$ in which f looks like

   $$(x^1, \dots, x^n, x^{n+1}, \dots, x^m) \mapsto (x^1, \dots, x^n).$$

   In the domain of the chart on M, the fibers of f are given by $x^1 = c^1, \dots, x^n = c^n$. Thus these coordinates serve as a flat chart for the foliation we want, perhaps after reversing the order of the coordinates to agree with Lee's definition on page 501.

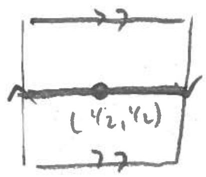
   (b) Let M be the Klein bottle, obtained as a quotient of the square in the usual way, and consider the foliation

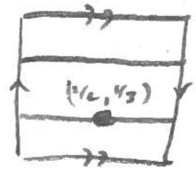
   

   Describe the leaf that passes through the central point $(\frac{1}{2}, \frac{1}{2})$, and the leaf that passes through $(\frac{1}{2}, \frac{1}{3})$. (An informal description is fine. It would be good to draw a picture. Notice that the left and right ends of the square have been glued with a half twist.)

   Prove that the leaves of this folation are not the fibers of any submersion $f\colon M \to S^1$. (Hint: I can think of several approaches. You might use the fact that a submersion has local sections, or that a submersion is locally of a certain form, or you could prove by hand that any smooth map $f\colon M \to S^1$ that is constant on the leaves of the foliation must fail to be a submersion at $(\frac{1}{2}, \frac{1}{2})\ldots$)

Solution: The leaf through $(\frac{1}{2}, \frac{1}{2})$ is a circle that "goes around" once. The leaf through $(\frac{1}{2}, \frac{1}{3})$ is a circle that goes around twice, also passing through $(\frac{1}{2}, \frac{2}{3})$.





For the claim that these leaves are not the fibers of a submersion, I'll prove it by hand, but other approaches are also fine. Let the usual coordinates on the square serve as coordinates on M near $(\frac{1}{2}, \frac{1}{2})$, and suppose that $f\colon M \to S^1$ is a smooth map that's constant on the leaves of the foliation. Then f is constant in the x-direction, so $\frac{\partial f}{\partial x} = 0$. Next, for $0 < h < \frac{1}{2}$ we have

$$f(\tfrac{1}{2}, \tfrac{1}{2} + h) = f(\tfrac{1}{2}, \tfrac{1}{2} - h),$$

so

$$\frac{\partial f}{\partial y}(\tfrac{1}{2}, \tfrac{1}{2}) = \lim_{h \to 0} \frac{f(\frac{1}{2}, \frac{1}{2} + h) - f(\frac{1}{2}, \frac{1}{2} - h)}{2h} = 0.$$

Thus the total derivative of f at $(\frac{1}{2}, \frac{1}{2})$ is zero, so f is not a submersion.

2. Adapted from Problem 20-13: Characterization of Lie algebra actions that correspond to transitive Lie group actions.

Suppose we have an action of a finite-dimensional Lie algebra $\mathfrak{g}$ on a smooth manifold M: that is, we have a homomorphism from $\mathfrak{g}$ to the Lie algebra of vector fields on M. Given $X \in \mathfrak{g}$, let $\hat{X}$ be the corresponding vector field on M. Say that the Lie algebra action is *transitive* if for every $p \in M$, the vectors $\hat{X}_p \in T_pM$ span T_pM as $X \in \mathfrak{g}$ varies.

We have seen that a right action of a Lie group G on a manifold M gives rise to an action of its Lie algebra $\mathfrak{g}$ on M, by taking

$$\hat{X}_p = D(\theta^{(p)})_1(X),$$

where $\theta^{(p)} \colon G \to M$ is the orbit map $g \mapsto p \cdot g$. In lecture this was our definition, and in Lee's book it's equation (20.8) on page 526.

Argue that the Lie algebra action is transitive if and only if the orbit maps $\theta^{(p)}$ are submersions. Prove that if M is connected, then the Lie group action is transitive if and only if the Lie algebra action is transitive in the sense above.

(Hints: On the midterm you proved that an equivariant map where the group acts transitively on the domain must have constant rank. And at one point you'll want to remember that a submersion is an open map.)

Solution: By definition, the Lie group action is transitive if and only if the derivative of $\theta^{(p)}$ is surjective at $1 \in G$ for all $p \in M$. If we let G act on itself by right multiplication, then $\theta^{(p)}$ is G-equivariant:

$$(p \cdot g) \cdot h = p \cdot (g \cdot h).$$

Since G acts transitively on itself, we see that the derivative of $\theta^{(p)}$ has constant rank, and in particular the derivative is surjective at 1 if and only if it's a submersion.

Now if the the Lie group action is transitive, then the orbit map $\theta^{(p)}$ is surjective for every $p \in M$, and a surjective map of constant rank is a submersion by the Global Rank Theorem (Theorem 4.14), so the Lie algebra action is transitive.

Conversely, if $\theta^{(p)}$ is a submersion, then it is an open map (by Proposition 4.28), so the orbit of p, which is the image of $\theta^{(p)}$, is open in M. Since M is a disjoint union of orbits, we see that every orbit is also closed. Because M is connected, this means there is only one orbit.