Solutions to Final Exam

Math 637

Wednesday, December 8, 2021

1. (a) Let \( f : M \to \tilde{N} \) be a submersion. Prove that the fibers of \( f \) give a foliation of \( M \).

   **Solution:** Let \( m = \dim M \) and \( n = \dim N \), so \( m \geq n \). By the Rank Theorem (Theorem 4.12), for every \( p \in M \) there are coordinate charts near \( p \) and \( f(p) \) in which \( f \) looks like

   \[
   (x^1, \ldots, x^n, x^{n+1}, \ldots, x^m) \mapsto (x^1, \ldots, x^n).
   \]

   In the domain of the chart on \( M \), the fibers of \( f \) are given by \( x^1 = c^1, \ldots, x^n = c^n \). Thus these coordinates serve as a flat chart for the foliation we want, perhaps after reversing the order of the coordinates to agree with Lee’s definition on page 501.

   (b) Let \( M \) be the Klein bottle, obtained as a quotient of the square in the usual way, and consider the foliation

   ![Diagram of a foliation](image)

   Describe the leaf that passes through the central point \((\frac{1}{2}, \frac{1}{2})\), and the leaf that passes through \((\frac{1}{2}, \frac{3}{2})\). (An informal description is fine. It would be good to draw a picture. Notice that the left and right ends of the square have been glued with a half twist.)

   Prove that the leaves of this foliation are not the fibers of any submersion \( f : M \to S^1 \). (Hint: I can think of several approaches. You might use the fact that a submersion has local sections, or...
that a submersion is locally of a certain form, or you could prove by hand that any smooth map \( f: M \to S^1 \) that is constant on the leaves of the foliation must fail to be a submersion at \( (\frac{1}{2}, \frac{1}{2}) \ldots \)

**Solution:** The leaf through \( (\frac{1}{2}, \frac{1}{2}) \) is a circle that "goes around" once. The leaf through \( (\frac{1}{2}, \frac{1}{3}) \) is a circle that goes around twice, also passing through \( (\frac{1}{2}, \frac{1}{3}) \).

For the claim that these leaves are not the fibers of a submersion, I'll prove it by hand, but other approaches are also fine. Let the usual coordinates on the square serve as coordinates on \( \tilde{M} \) near \( (\frac{1}{2}, \frac{1}{2}) \), and suppose that \( f: M \to S^1 \) is a smooth map that's constant on the leaves of the foliation. Then \( f \) is constant in the \( x \)-direction, so \( \frac{\partial f}{\partial x} = 0 \). Next, for \( 0 < h < \frac{1}{2} \) we have

\[
 f\left(\frac{1}{2}, \frac{1}{2} + h\right) = f\left(\frac{1}{2}, \frac{1}{2} - h\right),
\]

so

\[
 \frac{\partial f}{\partial y}\left(\frac{1}{2}, \frac{1}{2}\right) = \lim_{h \to 0} \frac{f\left(\frac{1}{2}, \frac{1}{2} + h\right) - f\left(\frac{1}{2}, \frac{1}{2} - h\right)}{2h} = 0.
\]

Thus the total derivative of \( f \) at \( (\frac{1}{2}, \frac{1}{2}) \) is zero, so \( f \) is not a submersion.

2. Adapted from Problem 20-13: Characterization of Lie algebra actions that correspond to transitive Lie group actions.

Suppose we have an action of a finite-dimensional Lie algebra \( \mathfrak{g} \) on a smooth manifold \( M \): that is, we have a homomorphism from \( \mathfrak{g} \) to the Lie algebra of vector fields on \( M \). Given \( X \in \mathfrak{g} \), let \( \tilde{X} \) be the corresponding vector field on \( M \). Say that the Lie algebra action is **transitive** if for every \( p \in M \), the vectors \( \tilde{X}_p \in T_pM \) span \( T_pM \) as \( X \in \mathfrak{g} \) varies.

We have seen that a right action of a Lie group \( G \) on a manifold \( M \) gives rise to an action of its Lie algebra \( \mathfrak{g} \) on \( M \), by taking

\[
 \tilde{X}_p = \tilde{D}(\theta(p))_1(X),
\]
where $\theta(p) : G \to M$ is the orbit map $g \mapsto p \cdot g$. In lecture this was our definition, and in Lee’s book it’s equation (20.8) on page 526.

Argue that the Lie algebra action is transitive if and only if the orbit maps $\theta(p)$ are submersions. Prove that if $M$ is connected, then the Lie group action is transitive if and only if the Lie algebra action is transitive in the sense above.

(Hints: On the midterm you proved that an equivariant map where the group acts transitively on the domain must have constant rank. And at one point you’ll want to remember that a submersion is an open map.)

**Solution:** By definition, the Lie group action is transitive if and only if the derivative of $\theta(p)$ is surjective at $1 \in G$ for all $p \in M$. If we let $G$ act on itself by right multiplication, then $\theta(p)$ is $G$-equivariant:

$$(p \cdot g) \cdot h = p \cdot (g \cdot h).$$

Since $G$ acts transitively on itself, we see that the derivative of $\theta(p)$ has constant rank, and in particular the derivative is surjective at 1 if and only if it’s a submersion.

Now if the the Lie group action is transitive, then the orbit map $\theta(p)$ is surjective for every $p \in M$, and a surjective map of constant rank is a submersion by the Global Rank Theorem (Theorem 4.14), so the Lie algebra action is transitive.

Conversely, if $\theta(p)$ is a submersion, then it is an open map (by Proposition 4.28), so the orbit of $p$, which is the image of $\theta(p)$, is open in $M$. Since $M$ is a disjoint union of orbits, we see that every orbit is also closed. Because $M$ is connected, this means there is only one orbit.