Solutions to Homework 1

Math 637

October 1, 2021

1-7. *Let N denote the north pole $(0, \ldots, 0, 1) \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$, and let S denote the south pole $(0, \ldots, 0, -1)$. Define the* stereographic projection *$\sigma \colon \mathbb{S}^n \setminus N \to \mathbb{R}^n$ by*

$$\sigma(x^1, \ldots, x^{n+1}) = \frac{(x^1, \ldots, x^n)}{1 - x^{n+1}}.$$

Let $\tilde{\sigma}(x) = \sigma(-x)$ for $x \in \mathbb{S}^n \setminus S$.

(a) *For any $x \in \mathbb{S}^n \setminus N$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$. Similarly, show that $\tilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace.*

The straight line through N and x can be parametrized by

$$\begin{aligned} \ell(t) &= (1-t)N + tx \\ &= (tx^1, \ldots, tx^n, 1 - t + tx^{n+1}). \end{aligned}$$

We see that $\ell(0) = N$ and $\ell(1) = x$. The last component of $\ell(t)$ will equal zero when $t = \frac{1}{1-x^{n+1}}$, and we have

$$\ell\left(\tfrac{1}{1-x^{n+1}}\right) = \left(\tfrac{x^1}{1-x^{n+1}}, \ldots, \tfrac{x^n}{1-x^{n+1}}, 0\right) = (\sigma(x), 0)$$

as desired. The claim about $\tilde{\sigma}$ and S is similar.

(b) *Show that σ is bijective, and*

$$\sigma^{-1}(u^1, \dots, u^n) = \frac{(2u^1, \dots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}.$$

We could check that the proposed σ^{-1} takes values in $\mathbb{S}^n \setminus N$ and that $\sigma \circ \sigma^{-1}$ and $\sigma^{-1} \circ \sigma$ are identity maps, although this is somewhat tedious. Here is another approach.

For $u \in \mathbb{R}^n$, the line through N and $(u, 0)$ will intersect $\mathbb{S}^n$ at two points, one of which is N and the other of which we want to find. Parametrize the line as

$$\ell(t) = (1 - t)N + t(u, 0)$$

We have

$$|\ell(t)|^2 = (1 - t)^2 + t^2 |u|^2,$$

because $|N|^2 = 1$ and $(u, 0)$ is orthogonal to N. To find the points where the line meets the sphere, we set $|\ell(t)|^2$ equal to 1, and after a little algebra we find that

$$-2t + (|u|^2 + 1)t^2 = 0.$$

One solution is $t = 0$, which gives the north pole. The other is

$$t = \frac{2}{|u|^2 + 1},$$

so we should take

$$\begin{aligned}
\sigma^{-1}(u^1, \dots, u^n) &= \ell\left(\frac{2}{|u|^2 + 1}\right) \\
&= \left(1 - \frac{2}{|u|^2 + 1}\right) N + \frac{2}{|u|^2 + 1}(u, 0) \\
&= \frac{|u|^2 - 1}{|u|^2 + 1} N + \frac{2}{|u|^2 + 1}(u, 0) \\
&= \frac{(2u^1, \dots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}.
\end{aligned}$$

(c) *Show that this smooth structure is the same as the one defined in Example 1.31.*

We need to show that for each of the $2n + 2$ coordinate charts in Example 1.4 and each of the the two stereographic charts, the transition maps in both directions are smooth. I will choose one chart from each atlas; the other choices are entirely similar.

One of the charts in Example 1.4 takes the northern hemisphere of S^n, that is, the set with $x^{n+1} > 0$, onto the unit ball in $\mathbb{R}^n$, via the map

$$(x^1, \ldots, x^n, x^{n+1}) \mapsto (x^1, \ldots, x^n). \tag{1}$$

The inverse takes

$$(x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^n, \sqrt{1 - (x^1)^2 - \cdots - (x^n)^2}). \tag{2}$$

The claim is that σ composed with (2) is a smooth map from the ball minus $(0, \ldots, 0)$ to $\mathbb{R}^n$ minus $(0, \ldots, 0)$, and that (1) composed with σ^{-1} is a smooth map the other way. We could write down formulas for these and observe that they are smooth. Or we could just say that σ extends to a smooth map $\mathbb{R}^{n+1} \setminus N \to \mathbb{R}^n$ given by the same formulas, and (1) extends to a smooth map $\mathbb{R}^{n+1} \to \mathbb{R}^n$, and we know that a composition of smooth maps between open subsets of $\mathbb{R}^n$ is again smooth.