All manifolds, maps, actions, etc. are smooth.

1. (a) Suppose that a Lie group $G$ acts on manifolds $M$ and $N$ on the left, and that $F: M \to N$ is an equivariant map, meaning that $F(g \cdot p) = g \cdot F(p)$ for all $g \in G$ and all $p \in M$. Prove that if $G$ acts transitively on $M$ then $F$ has constant rank.

(b) Let $G = \text{GL}_n(\mathbb{R})$, and let $F: G \to G$ be the map $F(A) = AA^\top$. Prove that $F$ is not a group homomorphism.

(c) Let $G$ act on the domain of $F$ by left multiplication. Describe a different action of $G$ on the codomain that makes $F$ into an equivariant map.

(d) Conclude that the orthogonal group $O(n) \subset G$ is a manifold.

(Hint: Lee’s Theorem 5.12 states that if a map has constant rank then its fibers are submanifolds. You may use this without proof.)

2. Let $G$ be a Lie group, and let $G_0 \subset G$ be the connected component of the identity. Because manifolds are locally path connected, connected components are the same as path components, which might be easier to work with.

(a) Prove that $G_0$ is a normal subgroup of $G$.

(b) Prove that any other connected component of $G$ is diffeomorphic to $G_0$.

(c) I was going to ask you to prove that if $U \subset G$ is a connected neighborhood of the identity, then the subgroup generated by $U$ is $G_0$. But I think the exam is already long enough.