Homework 1

Math 682

Due Friday, January 9, 2026

1. (a) Let $R = \mathbb{Z}[\sqrt{-3}]$. Sketch the lattice in the complex plane.

   (b) Prove that R is not integrally closed.
   Hint: Think about the minimal polynomial of
   $$\omega = e^{2\pi i/3} = \frac{-1+\sqrt{-3}}{2}.$$

   (c) Prove that 2 is irreducible in R by reasoning about norms, where $N(a+b\sqrt{-3}) = a^2+3b^2$. But prove that 2 is not prime by showing that the quotient ring $R/(2)$ is not an integral domain.

   (d) Same for $1+\sqrt{-3}$.

   (e) Prove that the ideal $\mathfrak{m} = (2, 1+\sqrt{-3})$ is maximal by showing that the quotient ring $R/\mathfrak{m}$ is a field. Prove that $\mathfrak{m}$ is the only prime ideal that contains 2 by reasoning about quotient rings. Same for $1+\sqrt{-3}$.

   (f) Prove that $\mathfrak{m}^2 = 2\mathfrak{m}$. Find the dimension of $\mathfrak{m}/\mathfrak{m}^2$ as a vector space over $R/\mathfrak{m}$. Prove that the principal ideals (2) and $(1+\sqrt{-3})$ are not powers of $\mathfrak{m}$, so they do not factor as products of primes.

   (g) Let $S = \mathbb{Z}[\omega]$. Use the fact that S is a principal ideal domain, and in fact a Euclidean domain (you may use this without proof), to prove that the Krull dimension of R is 1.
   Hint: Say "integral extension" and quote your favorite algebra book.
   (In the coordinate ring of an affine variety, $\mathfrak{m}/\mathfrak{m}^2$ was the Zariski cotangent space of the variety of the corresponding point. Because this $\mathfrak{m}/\mathfrak{m}^2$ is too big, we want to say that it's like a singular point.)

   (h) Let $\mathfrak{n} = \mathfrak{m}S$. Prove that $\mathfrak{n}$ is a principal ideal. Is it still prime? Describe the quotient ring $S/\mathfrak{n}$, which should contain $R/\mathfrak{m}$.

   (i) Find the dimension of $\mathfrak{n}/\mathfrak{n}^2$ as a vector space over $S/\mathfrak{n}$.

2. (a) Let $R = \mathbb{R}[x, y]/(y^2 + x^2 - x^3)$. Sketch the curve in $\mathbb{R}^2$.

   (b) Prove that R is not integrally closed.
   Hint: Let $z = y/x$ in $\text{frac}(R)$, and prove that $z^2 \in R$.

   (c) Prove that x is irreducible in R by reasoning about degrees. But prove that x is not prime by showing that the quotient ring $R/(x)$ is not an integral.

   (d) Same for y.

   (e) Prove that the ideal $\mathfrak{m} = (x, y)$ is maximal by showing that the quotient ring $R/\mathfrak{m}$ is a field. Prove that $\mathfrak{m}$ is the only prime ideal that contains x by reasoning about quotient rings. (But don't bother with y: it is contained in another maximal ideal, as you can see from the picture.)

   (f) Find the dimension of $\mathfrak{m}/\mathfrak{m}^2$ as a vector space over $R/\mathfrak{m}$. Prove that the principal ideal (x) is not a power of $\mathfrak{m}$, so it does not factor as a product of primes.

   (g) Let $S = \mathbb{R}[z]$. Describe the normalization map $\varphi \colon R \to S$ that sends y/x to z: where does it send x and y? Prove that the Krull dimension of R is 1.

   (h) Let $\mathfrak{n} = \varphi(\mathfrak{m})S$. Prove that $\mathfrak{n}$ is a principal ideal. Is it still prime? Describe the quotient ring $S/\mathfrak{n}$, which should contain $R/\mathfrak{m}$.

   (i) Find the dimension of $\mathfrak{n}/\mathfrak{n}^2$ as a vector space over $S/\mathfrak{n}$.

3. Optional: I might have preferred to work with $\mathbb{R}[x, y]/(y^2 - x^2 - x^3)$ because the picture is prettier, but then the ideal $\mathfrak{m} = (x, y)$ splits in the normalization rather than being inert, so the analogy with $\mathbb{Z}[\sqrt{-3}]$ is not as good...

   Can you find a square-free integer D with $D \equiv 1 \pmod 4$ such that the maximal ideal $\mathfrak{m} = (2, 1 + \sqrt{D})$ in $R = \mathbb{Z}[\sqrt{D}]$ splits when you extend to $S = \mathbb{Z}[\frac{1+\sqrt{D}}{2}]$?