Messy Lagrange Multipliers Example

Math 212

Thursday, September 27, 2012

**Problem:** Find the points on the ellipse \( x^2 + xy + 2y^2 = 1 \) farthest from the origin using the method of Lagrange multipliers.

**Solution:** We maximize the function \( f(x, y) = x^2 + y^2 \) with the constraint \( g(x, y) = x^2 + xy + 2y^2 = 1 \). We have \( \nabla f = \langle 2x, 2y \rangle \) and \( \nabla g = \langle 2x+y, x+4y \rangle \), so we want to solve the equations

\[
\begin{align*}
2x &= \lambda(2x + y) \quad (1) \\
2y &= \lambda(x + 4y) \quad (2) \\
x^2 + xy + 2y^2 &= 1. \quad (3)
\end{align*}
\]

First observe that for any given \( \lambda \), equations (1) and (2) describe two lines through the origin: solving both for \( y \) we have

\[
y = \frac{2 - 2\lambda}{\lambda} x \quad y = \frac{\lambda}{2 - 4\lambda} x. \quad (4)
\]

If these lines have different slopes then they intersect only at the origin, which contradicts (3), so they must have the same slope:

\[
\frac{2 - 2\lambda}{\lambda} = \frac{\lambda}{2 - 4\lambda}.
\]

Clearing denominators we have

\[
\begin{align*}
4 - 12\lambda + 8\lambda^2 &= \lambda^2 \\
7\lambda^2 - 12\lambda + 2 &= 0 \\
\lambda &= \frac{6 \pm 2\sqrt{2}}{7}. \quad (5)
\end{align*}
\]

Note that we found this on Thursday in a more complicated way.
We can use Sharrin/Andrew/Euler’s trick that we saw in the $xyz = 8$ problem on Thursday to rule out one of the roots (5); this is not strictly necessary, but it cuts a messy calculation in half. Multiply equation (1) by $x$ and equation (2) by $y$ and add them up to get

$$2x^2 + 2y^2 = \lambda(2x^2 + 2xy + 4y^2).$$

Divide through by 2 and use equation (3) to get

$$x^2 + y^2 = \lambda.$$

Thus since we want the points farthest from the origin, we only want the larger root

$$\lambda = \frac{6 + 2\sqrt{2}}{7}.$$

Now the two lines (4) become

$$y = (1 - \sqrt{2})x.$$

Plugging this into (3) we get

$$(8 - 5\sqrt{2})x^2 = 1,$$

so

$$x = \pm \sqrt{\frac{8 + 5\sqrt{2}}{14}},$$

and thus

$$y = (1 - \sqrt{2})x = \mp \sqrt{\frac{4 - \sqrt{2}}{14}}.$$

Note that the plus-or-minus became minus-or-plus because $1 - \sqrt{2}$ is negative.