First Midterm

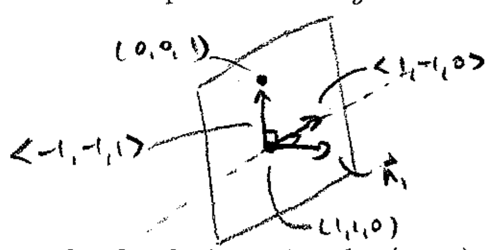
Name: Solutions

February 19, 2013

1. (a) Write an equation for the plane containing the line $x = 1 + t$, $y = 1 - t$, $z = 0$ and the point $(0, 0, 1)$. Check that the line and the point satisfy the equation you come up with.
Solution: The direction vector of the line is $\langle 1, -1, 0\rangle$. The point $(1, 1, 0)$ is on the line, and the vector from that point to $(0, 0, 1)$ is $\langle -1, -1, 1\rangle$. To get the normal vector to the plane we cross these two vectors:

$$\mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ -1 & -1 & 1 \end{vmatrix} = \langle -1, -1, -2\rangle.$$

Thus an equation for the plane is $-x - y - 2z = -2$.



Indeed we have $-0 - 0 - 2 \cdot 1 = -1$ and $-(1 + t) - (1 - t) - 2 \cdot 0 = -2$.

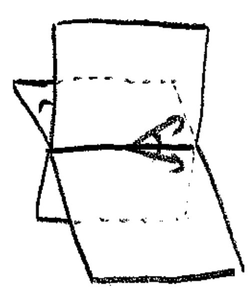
(b) Write an equation for the plane containing the same line and the point $(2, 2, 1)$.
Solution: The vector from $(1, 1, 0)$ to $(2, 2, 1)$ is $\langle 1, 1, 1\rangle$, so the normal vector to the plane is

$$\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \langle -1, -1, 2\rangle.$$

Thus an equation for the plane is $-x - y + 2z = -2$.

(c) Find cosine of the angle between the planes in parts (a) and (b).
Solution: The angle between the planes is the angle between their normal vectors:

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{-2}{\sqrt{6}\sqrt{6}} = -\frac{1}{3}.$$



2. The shape of a power line hanging between two telephone poles (or any other hanging cable or chain) is not a parabola as you might guess, but a *catenary*, described by the equation

$$y = \frac{e^x + e^{-x}}{2}.$$

Parametrize this as

$$\mathbf{r}(t) = \langle t, \tfrac{e^t+e^{-t}}{2} \rangle$$

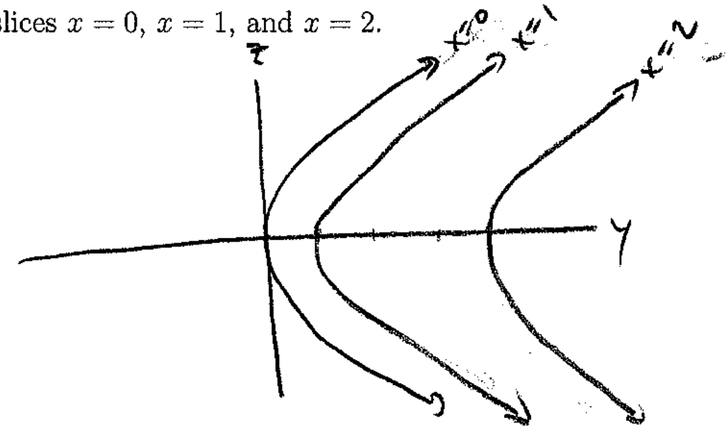
and find the length of the curve between $t = -1$ and $t = 1$. Hint: The thing under the square root can be factored; look at the thing you expanded out to get an idea for how to factor it.

Solution: We have $\mathbf{r}'(t) = \langle 1, \frac{e^t-e^{-t}}{2} \rangle$, so the arc length is

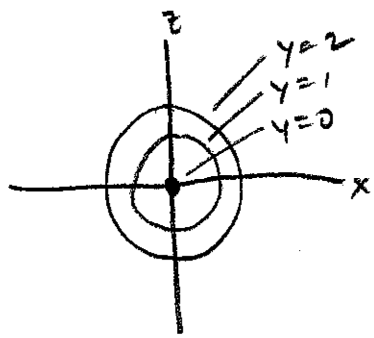
$$\begin{aligned}
\int_{-1}^{1} |\mathbf{r}'(t)|\, dt &= \int_{-1}^{1} \sqrt{1 + \left(\frac{e^t - e^{-t}}{2}\right)^2}\, dt \\
&= \int_{-1}^{1} \sqrt{1 + \frac{e^{2t} - 2 + e^{-2t}}{4}}\, dt \\
&= \int_{-1}^{1} \sqrt{\frac{e^{2t} + 2 + e^{-2t}}{4}}\, dt \\
&= \int_{-1}^{1} \sqrt{\left(\frac{e^t + e^{-t}}{2}\right)^2}\, dt \\
&= \int_{-1}^{1} \frac{e^t + e^{-t}}{2}\, dt \\
&= \left[\frac{e^t - e^{-t}}{2}\right]_{-1}^{1} \\
&= e - 1/e = 2.3504023872876\ldots
\end{aligned}$$

3. Consider the surface $y = x^2 + z^2$.

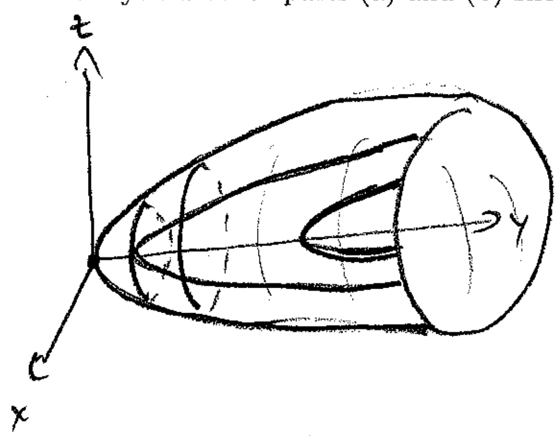
(a) Sketch the slices $x = 0$, $x = 1$, and $x = 2$.
Solution:



(b) Sketch the slices $y = 0$, $y = 1$, and $y = 2$.
Solution:



(c) Sketch the surface. The slices you drew in parts (a) and (b) should appear.
Solution:



4. Find the limit, or show that it does not exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.

Solution: If we approach the origin along the line $y = x$ we find that the limit is $\frac{1}{2}$, but if we approach along the line $y = -x$ we find that the limit is $-\frac{1}{2}$; thus the limit does not exist.

(b) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$.

Solution: Change to polar coordinates, setting $x = r\cos\theta$ and $y = r\sin\theta$; then

$$\frac{x^2y^2}{x^2+y^2} = \frac{r^4\cos^2\theta\sin^2\theta}{r^2} = r^2\cos^2\theta\sin^2\theta.$$

Since $0 \le \cos^2\theta\sin^2\theta \le 1$, we see that

$$0 \le r^2\cos^2\theta\sin^2\theta \le r^2.$$

Since the right-hand side $r^2 \to 0$ as $(x,y) \to (0,0)$, we see that the middle term $r^2\cos^2\theta\sin^2\theta \to 0$ as well.

5. Find the maximum and minimum values of the function $f(x, y) = x^2 + x + y^2 - y$ on the disc $x^2 + y^2 \leq 1$.

Solution: First we look for maxes and mins on the interior of the disc, so we set $f_x = 0$ and $f_y = 0$:

$$2x + 1 = 0 \qquad\qquad 2y - 1 = 0$$

Thus $x = -\frac{1}{2}$ and and $y = \frac{1}{2}$. We have $f(-\frac{1}{2}, \frac{1}{2}) = -\frac{1}{2}$.

Next we look for maxes and mins on the boundary of the disc, which we parametrize as $x = \cos t$, $y = \sin t$. Then we have

$$x^2 + x + y^2 - y = 1 + \cos t - \sin t.$$

Taking the derivative with respect to t and setting it equal to zero we get

$$\begin{aligned} -\sin t - \cos t &= 0 \\ -\cos t &= \sin t \\ -1 &= \frac{\sin t}{\cos t} = \tan t \end{aligned}$$

so $t = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$. In the first case we have

$$f(\cos \tfrac{3\pi}{4}, \sin \tfrac{3\pi}{4}) = f(-\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}) = 1 - \sqrt{2}.$$

In the second case we have

$$f(\cos \tfrac{7\pi}{4}, \sin \tfrac{7\pi}{4}) = f(\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}) = 1 + \sqrt{2}.$$

Since $-\frac{1}{2} < 1 - \sqrt{2} < 1 + \sqrt{2}$, the minimum is $-\frac{1}{2}$ and the maximum is $1 + \sqrt{2}$.

6. Use linear approximation to find the approximate value of $\sqrt{(.9)(2.9)(3.1)}$.
Hint: The true value is 2.84446831587205...
Solution: Set $f(x,y,z) = \sqrt{xyz}$. Then the partial derivatives are

$$f_x(x,y,z) = \frac{\sqrt{yz}}{2\sqrt{x}} \qquad f_y(x,y,z) = \frac{\sqrt{xz}}{2\sqrt{y}} \qquad f_z(x,y,z) = \frac{\sqrt{xy}}{2\sqrt{z}}.$$

In particular we have

$$f(1,3,3) = 3 \qquad f_x(1,3,3) = \frac{3}{2} = 1.5 \qquad f_y(1,3,3) = \frac{1}{2} = .5 \qquad f_z(1,3,3) = \frac{1}{2} = .5.$$

Then linear approximation says that

$$\begin{aligned} f(1-.1, 3-.1, 3+.1) &\approx f(1,3,3) - .1\cdot f_x(1,3,3) - .1\cdot f_y(1,3,3) + .1\cdot f_z(1,3,3) \\ &= 3 - .15 - .05 + .05 = 2.85, \end{aligned}$$

Which is within 0.2% of the true value.