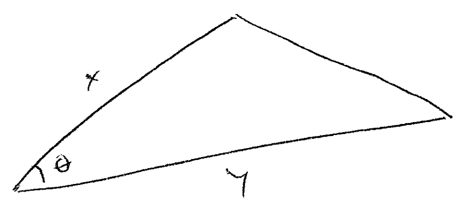
Second Midterm

Name: Solutions

April 2, 2013

1. (a) The lengths of the two sides of a triangle are x and y, and θ is the angle between the two sides. Draw a picture.
Solution:



(b) The area of the triangle is $A = \frac{1}{2}xy\sin\theta$. If x is increasing at a rate of 2 inches per second, y is decreasing at a rate of 2 inches per second and θ is increasing at a rate of 0.1 radians per second, how fast is the area changing at the instant when $x = y = 3$ feet, and $\theta = \pi/3$ radians?
Solution: We use the chain rule:

$$\begin{aligned}
\frac{dA}{dt} &= \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} + \frac{\partial A}{\partial \theta}\frac{d\theta}{dt} \\
&= \tfrac{1}{2}y\sin\theta \cdot \tfrac{dx}{dt} + \tfrac{1}{2}x\sin\theta \cdot \tfrac{dy}{dt} + \tfrac{1}{2}xy\cos\theta \cdot \tfrac{d\theta}{dt} \\
&= \tfrac{1}{2}\cdot 3 \cdot \tfrac{\sqrt{3}}{2} \cdot \tfrac{2}{12} + \tfrac{1}{2}\cdot 3 \cdot \tfrac{\sqrt{3}}{2} \cdot \left(-\tfrac{2}{12}\right) + \tfrac{1}{2}\cdot 3 \cdot 3 \cdot \tfrac{1}{2} \cdot 0.1 \\
&= \tfrac{9}{40}\ \tfrac{\text{ft}^2}{\text{sec}} = 0.225\ \tfrac{\text{ft}^2}{\text{sec}} = 32.4\ \tfrac{\text{in}^2}{\text{sec}}
\end{aligned}$$

2. Write an equation for the tangent plane to the surface

$$x^3 + y^3 + z^3 + 1 = (x + y + z + 1)^3$$

at the point $(2, -2, -1)$. Hint: Don't waste time multiplying out the right-hand side.

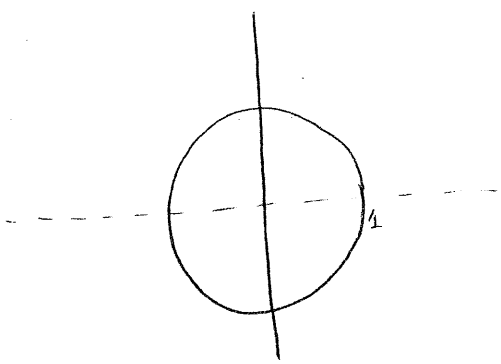
Solution: Let $f(x, y, z) = x^3 + y^3 + z^3 + 1 - (x + y + z + 1)^3$. Then we're talking about the level surface $f(x, y, z) = 0$, so the normal vector to the tangent plane is $\nabla f(2, -2, -1)$. We have

$$f_x(x, y, z) = 3x^2 - 3(x + y + z + 1)^2$$
$$f_y(x, y, z) = 3y^2 - 3(x + y + z + 1)^2$$
$$f_z(x, y, z) = 3z^2 - 3(x + y + z + 1)^2,$$

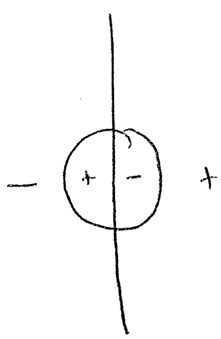
so $\nabla f(2, -2, 1) = \langle 12, 12, 3\rangle$. Thus the equation of the plane is $12x + 12y + 3z = -3$, or $4x + 4y + z = -1$ if you prefer.

3. Let $f(x, y) = x(x^2 + y^2 - 1)$.

   (a) Sketch the level set $f(x, y) = 0$.
   Solution: It has two components: the line $x = 0$ and the circle $x^2 + y^2 = 1$.



   (b) Find $f(2, 0)$. From this piece of information and your sketch above, guess where the function is positive and negative; indicate it with plusses and minuses on your sketch.
   Solution: We find that $f(2, 0) = 6$, so the function is positive on the right, and we guess that the sign alternates from one region to the next:



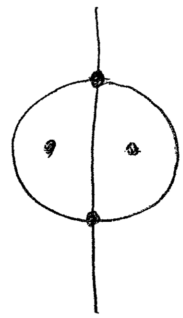
   (c) Find the four critical points of f. Add them to your sketch.

Solution: We have $f = x^3 + xy^2 - x$, so

$$f_x = 3x^2 + y^2 - 1 \qquad\qquad f_y = 2xy.$$

Since $f_y = 0$, we must have $x = 0$ or $y = 0$. If $x = 0$ then $f_x = 0$ implies $y = \pm 1$. If $y = 0$ then $f_x = 0$ implies $x = \pm 1/\sqrt{3}$. Thus the four points are

$$(0, \pm 1) \qquad\qquad (\pm\tfrac{1}{\sqrt{3}}, 0).$$



(d) Decide whether each critical point is a local minimum, a local maximum, or a saddle point.
Solution: We have

$$f_{xx} = 6x \qquad\qquad f_{xy} = 2y \qquad\qquad f_{yy} = 2x$$

so the discriminant is

$$\Delta(x, y) = \det \begin{pmatrix} 6x & 2y \\ 2y & 2x \end{pmatrix} = 12x^2 - 4y^2.$$

Then $\Delta(0, \pm 1) = -4$, so those two are saddle points, and $\Delta(\pm\frac{1}{\sqrt{3}}, 0) = 4$, so those are local minima or local maxima; moreover $f_{xx}(\pm\frac{1}{\sqrt{3}}, 0) = \pm 2\sqrt{3}$, so $(\frac{1}{\sqrt{3}}, 0)$ is a local minimum and $(-\frac{1}{\sqrt{3}}, 0)$ is a local maximum.

(e) Find the maximum value of f on the circle $x^2 + y^2 = 4$ using Lagrange multipliers.
Solution: Let $g(x, y) = x^2 + y^2$, so we wish to solve $\nabla f = \lambda \nabla g$ and $g(x, y) = 4$, that is,

$$3x^2 + y^2 - 1 = 2\lambda x \tag{1}$$
$$2xy = 2\lambda y \tag{2}$$
$$x^2 + y^2 = 4. \tag{3}$$

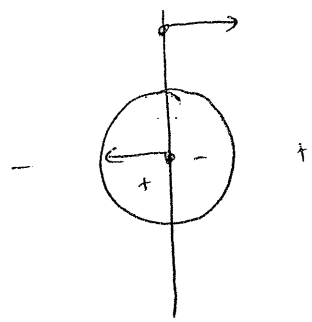
From (2) we see that $x = \lambda$ or $y = 0$. If $y = 0$ then from (3) we see that $x = \pm 2$. If $x = \lambda$ then (1) becomes $3x^2 + y^2 - 1 = 2x^2$, so $x^2 + y^2 = 1$, which contradicts (3). Thus the only points of interest are $(\pm 2, 0)$. We have already found that $f(2, 0) = 6$, and similarly we find that $f(-2, 0) = -6$, so the maximum value on the circle is 6.

(f) Find the maximum value of f on the disc $x^2+y^2 \leq 4$. Does it occur in the interior or on the boundary (or both)?
Solution: We just found that the maximum value on the boundary is 6. If the maximum occurs in the interior it occurs at the local maximum $(-\frac{1}{\sqrt{3}}, 0)$, but

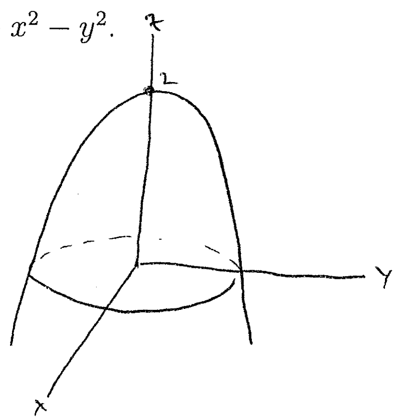
$$f(-\tfrac{1}{\sqrt{3}}, 0) = \tfrac{2}{3\sqrt{3}}$$

which is much less than 6. Thus 6 is the maximum on the whole disc.

(g) In what direction does f increase most steeply at the origin? At the point $(0, 2)$? Add the appropriate vectors to your sketch.
Solution: We have $\nabla f(0,0) = \langle -1, 0\rangle$, so the steepest increase is gotten by going straight to the left. We have $\nabla f(0,2) = \langle 3, 0\rangle$, so the steepest increase is gotten by going straight to the right.



4. (a) Sketch the surface $z = 2 - x^2 - y^2$.
Solution:



(b) Find the volume of the region that is below the surface and above the xy-plane. Do the integral in polar coordinates.
Solution: The intersection of the paraboloid and the xy-plane is the circle $x^2 + y^2 = 2$, which has a radius of $\sqrt{2}$. In polar coordinates the equation of the

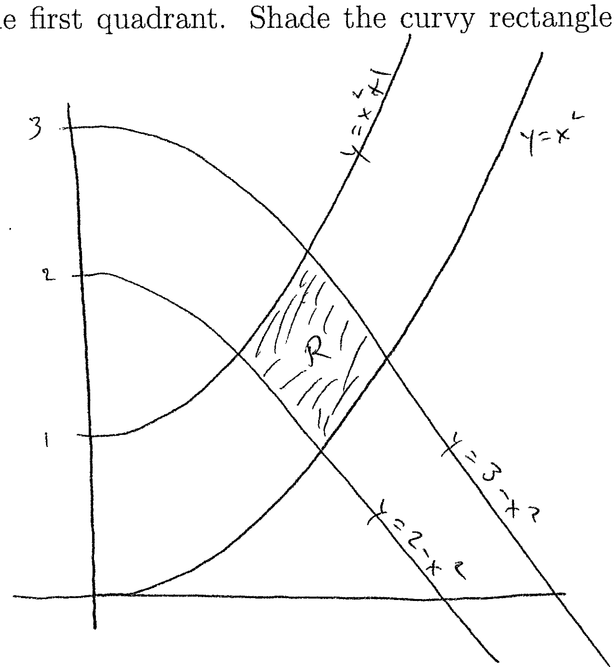
paraboloid becomes $z = 2 - r^2$, and the volume is

$$\int_0^{2\pi}\int_0^{\sqrt{2}} (2-r^2)\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} (2r - r^3)\, dr$$
$$= 2\pi \left[r^2 - \frac{1}{4} r^4 \right]_0^{\sqrt{2}}$$
$$= 2\pi(2-1) = 2\pi.$$

5. This problem is about Jacobians (section 14.9).

(a) Sketch the portions of the curves $y = x^2$, $y = x^2 + 1$, $y = 2 - x^2$, and $y = 3 - x^2$ that lie in the first quadrant. Shade the curvy rectangle that they bound and label it R.

Solution:

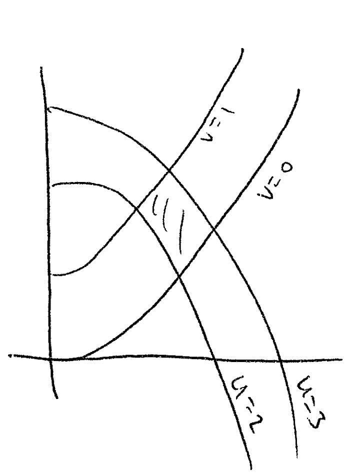


(b) We wish to find the center of mass of this region R. You can find the y-coordinate by symmetry, without doing any integrals. What is it?

Solution: Since R is symmetric about the line $y = 3/2$, that's the y-coordinate.

(c) Let $u = y + x^2$ and $v = y - x^2$. What do the four curves in part (a) become in these new coordinates? Indicate this on your sketch, or draw a new one if the old one is becoming a mess.

Solution: The curves become $v = 0$, $v = 1$, $u = 2$, and $u = 3$, as shown:



(d) Solve for x and y in terms of u and v. Hint: Consider $u+v$ and $u-v$.
Solution: Taking the hint, we have

$$u + v = 2y \qquad\qquad u - v = 2x^2,$$

so

$$x = \sqrt{\frac{u-v}{2}} \qquad\qquad y = \frac{u+v}{2}.$$

(e) Evaluate the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}.$$

Don't be discouraged by the profusion of twos.
Solution: We have $x = \frac{1}{\sqrt{2}}(u-v)^{1/2}$, so

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{1}{2\sqrt{2}}(u-v)^{-1/2} & -\frac{1}{2\sqrt{2}}(u-v)^{-1/2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2\sqrt{2}}(u-v)^{-1/2}.$$

(f) When you take an integral

$$\iint_R \text{something } dx\, dy$$

and change variables to u and v, what will happen to the bounds? What will happen to $dx\, dy$?
Solution: The integral will become

$$\int_2^3 \int_0^1 (\text{something else}) \; \frac{1}{2\sqrt{2}}(u-v)^{-1/2}\, dv\, du.$$

(g) Find $\iint_R x \; dx\, dy$. Hint: There's a lot of cancellation.
Solution:

$$\begin{aligned}\iint_R x \; dx\, dy &= \int_2^3 \int_0^1 \tfrac{1}{\sqrt{2}}(u-v)^{1/2} \; \tfrac{1}{2\sqrt{2}}(u-v)^{-1/2}\, dv\, du \\ &= \int_2^3 \int_0^1 \tfrac{1}{4}\, dv\, du \\ &= \tfrac{1}{4}.\end{aligned}$$

(h) Find the area of the region, i.e. find $\iint_R 1 \, dx\, dy$.
Hint: Notice that $\int (u-v)^{-1/2}\, dv = -2(u-v)^{1/2} + C$. The integral is doable but

the answer isn't very pretty.
Solution:

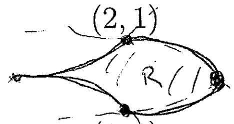
$$\iint_R dx\,dy = \int_2^3 \int_0^1 \tfrac{1}{2\sqrt{2}}(u-v)^{-1/2}\,dv\,du$$
$$= \int_2^3 \left[-\tfrac{1}{\sqrt{2}}(u-v)^{1/2}\right]_0^1 du$$
$$= \int_2^3 \left(\tfrac{1}{\sqrt{2}}u^{1/2} - \tfrac{1}{\sqrt{2}}(u-1)^{1/2}\right) du$$
$$= \left[\tfrac{\sqrt{2}}{3}u^{3/2} - \tfrac{\sqrt{2}}{3}(u-1)^{3/2}\right]_2^3$$
$$= \left(\tfrac{\sqrt{2}}{3}3^{3/2} - \tfrac{\sqrt{2}}{3}2^{3/2}\right) - \left(\tfrac{\sqrt{2}}{3}2^{3/2} - \tfrac{\sqrt{2}}{3}1^{3/2}\right)$$
$$= \sqrt{6} - \tfrac{8}{3} + \tfrac{\sqrt{2}}{3} \approx .254.$$

(i) Find the x-coordinate of the center of mass. You don't need to simplify it.
Solution:

$$\frac{\frac{1}{4}}{\sqrt{6} - \frac{8}{3} + \frac{\sqrt{2}}{3}} \approx 0.983.$$

If you're curious, the vertices of the curvy rectangle are at

$(2,1)$

$(\sqrt{\tfrac{1}{2}}, \tfrac{3}{2}) \approx (0.707, 1.5)$  $(\sqrt{\tfrac{3}{2}}, \tfrac{3}{2}) \approx (1.225, 1.5)$,

$(1,1)$

so the center of mass is slightly to the right of the middle pair of vertices.