Solutions to Homework 2

Math 487

September 16, 2013

III.3.3 First we produce an interpretation in which all three formulas hold. (This is more than was required.) We take the domain set A to be the one-element set $\{0\}$. We interpret the relation symbol as $\mathfrak{a}(P) = A$, that is, $P0$ is true. We must necessarily interpret the function symbol as $\mathfrak{a}(f)(0) = 0$, and take the assignment to be $\beta(v_i) = 0$ for all i. By inspection this satisfies all three formulas.

Next we produce an interpretation in which all three formulas fail. (Again this is more than was required.) We take the domain set A to be the two-element set $\{0, 1\}$. We interpret the relation symbol as $\mathfrak{a}(P) = \emptyset$, that is, $P0$ is false and $P1$ is false. We interpret the function symbol as $\mathfrak{a}(f)(a) = 1$ for all $a \in A$. We take the assignment to be $\beta(v_i) = 0$ for all i. Clearly the first formula is not satisfied. For the second formula, there is no v_0 such that for all v_1 one has $f(v_0, v_1) = v_1$, since for any v_0 we could take $v_1 = 0$ and get $f(v_0, 0) \neq 0$, so the second formula is not satisfied. The third formula is not satisfied because Pv_0 is always false.

III.4.14 We label the group axioms as follows:

$$\begin{aligned}
\varphi_{\text{ass}} &= \forall v_0 \forall v_1 \forall v_2\ (v_0 \circ v_1) \circ v_2 \equiv v_0 \circ (v_1 \circ v_2) && \text{(associativity)}\\
\varphi_{\text{id}} &= \forall v_0\ v_0 \circ e \equiv v_0 && \text{(identity)}\\
\varphi_{\text{inv}} &= \forall v_0 \exists v_1\ v_0 \circ v_1 \equiv e && \text{(inverse).}
\end{aligned}$$

For a structure in which φ_{ass} and φ_{id} hold but φ_{inv} fails, take $A = \mathbb{N}$, take $\circ$ to be addition, and take $e = 0$.

For a structure in which φ_{ass} and φ_{inv} hold but φ_{id} fails, take $A = \mathbb{Z}$, take $\circ$ to be addition, and take $e = 1$.

For a structure in which φ_{inv} and φ_{id} hold but φ_{ass} fails, take the unit octonions $\{\pm 1, \pm e_1, \ldots, \pm e_7\}$. Define $\circ$ according to the table

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

and extend it to $-e_i$ in the obvious way. (I have lifted the table from John Baez's wonderful expository paper *The Octonions*, which is available on the web.) The identity is 1. The inverse of e_i is $-e_i$. To see that $\circ$ is not associative, we calculate

$$\begin{aligned}(e_1 \circ e_2) \circ e_3 &= e_4 \circ e_3 = -e_6 \\ e_1 \circ (e_2 \circ e_3) &= e_1 \circ e_5 = e_6.\end{aligned}$$

There are certainly simpler examples than this one, but they are less important.

IV.3.6(a2)

1.	Γ		$\neg\neg\varphi$	premise
2.	Γ	$\neg\varphi$	$\neg\varphi$	(Assm)
3.	Γ	$\neg\varphi$	$\neg\neg\varphi$	(Ant) applied to 1.
4.	Γ		φ	(Ctr) applied to 2. and 3.