VI.4.10 Let $P = \{2, 3, 5, 7, 11, \ldots\}$ be the set of prime numbers. Let $S = \{0, 1, +, \times\}$, and let $\mathcal{N}$ be the standard $S$-structure on $\mathbb{N}$. Given an $S$-structure $\mathfrak{A} \models \text{Th}(\mathcal{N})$ and an element $a \in A$, we define
\[
\text{div}(a, \mathfrak{A}) := \{ p \in P \mid \text{there is a } b \in A \text{ such that } a = b +^\mathfrak{A} \cdots +^\mathfrak{A} b \},
\]
i.e. the set of primes dividing $a$ according to $\mathfrak{A}$.

First we argue that for any subset $Q \subset P$ there is a countable $S$-structure $\mathfrak{A} \models \text{Th}(\mathcal{N})$ and an element $a \in A$ with $\text{div}(a, \mathfrak{A}) = Q$. For each $p \in P$, consider the formula
\[
\varphi_p := \exists v_1 \; v_0 = v_1 + \cdots + v_1,
\]
which says that $v_0$ is divisible by $p$. Given a subset $Q \subset P$, let
\[
\Phi_Q = \text{Th}(\mathcal{N}) \cup \{ \varphi_p \mid p \in Q \} \cup \{ \neg \varphi_p \mid p \in P \setminus Q \}.
\]
Every finite subset $\Psi \subset \Phi_Q$ is satisfiable, as follows. We have
\[
\Psi \subset \text{Th}(\mathcal{N}) \cup \{ \varphi_{p_1}, \ldots, \varphi_{p_m} \} \cup \{ \neg \varphi_{p_{m+1}}, \ldots, \neg \varphi_{p_{m+n}} \}
\]
for some distinct primes $p_1, \ldots, p_{m+n}$. The interpretation consisting of the structure $\mathfrak{A}$ and the assignment $\beta(v_i) = p_1 \cdots p_m \in \mathbb{N}$ satisfies $\Psi$, as claimed. Thus every finite subset of $\Phi_Q$ is consistent, so $\Phi_Q$ is consistent, so there is a countable interpretation $(\mathfrak{A}, \beta') \models \Phi_Q$. Now $a := \beta'(v_0) \in A$ has the required property.\(^1\)

\(^1\)Note that $Q$ may be empty; in that case our proof says to take $\beta(v_i) = 1$ for all finite subsets $\Psi \subset \Phi_Q$, which is no problem. Thus there is a non-standard model of arithmetic containing an infinite element $a$ which is not divisible by any finite element!
Next we argue that there are uncountably many non-isomorphic countable models of \(\text{Th}(\mathcal{N})\). Suppose that \(A_0, A_1, A_2, \ldots\) are countable models of \(\text{Th}(\mathcal{N})\) with \(A_i \not\cong A_j\) for \(i \neq j\); we will produce a countable model \(A \models \text{Th}(\mathcal{N})\) which is not isomorphic to any \(A_i\). Each \(A_i\) is countable, and a countable union of countable sets is countable,\(^2\) so only countably many subsets of \(P\) are of the form \(\text{div}(a, A_i)\) for some \(i\) and some \(a \in A_i\). But \(P\) is infinite, so the set of subsets of \(P\) is uncountable as we showed in exercise I.1.5. Hence there is a \(Q \subset P\) which is not of the form \(\text{div}(a, A_i)\) for some \(i\) and some \(a \in A_i\). But we saw above that there is a countable model \(A \models \text{Th}(\mathcal{N})\) and an element \(a \in A\) with \(\text{div}(a, A) = Q\); hence \(A\) is not isomorphic to any \(A_i\), as required.

**Problem on ordinals:**

(a) Rewrite the formula \(x = y \cup \{y\}\) using only \(\in\).

\[
\forall z \, (z \in x \leftrightarrow (z \in y \lor z \equiv y))
\]

(b) Write a formula using only \(\in\) that means “\(x\) is an ordinal.”

\[
\forall y \, (y \in x \rightarrow \forall z \, (z \in y \rightarrow z \in x))
\]

(c) Write a formula using only \(\in\) that means “\(x\) is a finite ordinal.” There are many equally good answers, but here is one:

\[
\begin{align*}
&\text{\(x\) is an ordinal} \\
&\forall y \left(\left(\forall z \, (z \in y \rightarrow \forall w \, (w \in z \rightarrow w \in y))\right) \land \\
&\left(\forall y \, (y \subset x) \land \forall z \, (z \in x) \land \forall w \, (w \in z \rightarrow w \in y)) \rightarrow \\
&\left(\forall y \, (y = \emptyset) \land \forall z \, (z \equiv y) \land \forall w \, (w \in y \leftrightarrow (w \in z \land w \equiv y))\right)\right) \\
\end{align*}
\]

This could be optimized in various ways, at the cost of being even more opaque; for example the two occurrences of \(\forall y\) could be replaced with one \(\forall y\) before the first parenthesis, or in the second line we could combine the two \(\forall z \, (z \in y \rightarrow \cdots)\) pieces into one:

\[
\forall z \, (z \in y \rightarrow (z \in x \land \forall w \, (w \in z \rightarrow w \in y))
\]

which says \(y\) is an ordinal contained in \(x\).

\(^2\)Here we have used the axiom of choice, but with more care (i.e. more fuss) we could have avoided doing so.