Second Midterm

Algebraic Topology M3P21/M4P21

Due Tuesday, March 23, 2010

Same policy as last time: work together but write alone, in pencil, double spaced (or \TeX).

Recall that a topological group is a topological space $G$ with a distinguished point $1 \in G$ and continuous maps $m : G \times G \to G$ and $i : G \to G$ satisfying the group axioms:

$$m(g, m(h, k)) = m(m(g, h), k)$$
$$m(g, 1) = m(1, g) = g$$
$$m(i(g), g) = m(g, i(g)) = 1.$$

Of course $m(g, h)$ is usually denoted $gh$ and $i(g)$ is usually denoted $g^{-1}$. Examples:

- The general linear group $GL_n \mathbb{R}$ of $n \times n$ invertible matrices.
- The special linear group $SL_n \mathbb{R}$ of matrices $A$ with $\det A = 1$.
- The orthogonal group $O(n)$ of matrices $A$ with $A^\top = A^{-1}$. These are the automorphisms of $\mathbb{R}^n$ that preserve the inner product $\langle x, y \rangle = x_1y_1 + \cdots + x_ny_n = x^\top y$, for
  $$(Ax)^\top (Ay) = x^\top A^\top Ay = x^\top A^{-1}Ay = x^\top y.$$
- The special orthogonal group $SO(n) = O(n) \cap SL_n \mathbb{R}$, also known as the group of rotations. Later we will see that $\pi_1(SO(3)) = \mathbb{Z}/2$.
- Their complex analogues $GL_n \mathbb{C}$, $SL_n \mathbb{C}$, $U(n)$, and $SU(n)$. For the unitary group $U(n)$ the condition is $A^\top = A^{-1}$, so $A$ preserves the Hermitian product $\langle z, w \rangle = \bar{z}^\top w$ on $\mathbb{C}^n$.

1. Show that the inclusion $O(n) \hookrightarrow GL_n \mathbb{R}$ is a homotopy equivalence. Hint: The columns of a matrix $A$ form a basis for $\mathbb{R}^n$ if and only if $A \in GL_n \mathbb{R}$. They form an orthonormal basis if and only if $A \in O(n)$. The Gram-Schmidt algorithm gives a deformation retraction of $GL_n \mathbb{R}$ onto $O(n)$.

Solution: Let $A_1, \ldots, A_n$ be the columns of a matrix $A$:

$$A = \begin{pmatrix} A_1 & \ldots & A_n \end{pmatrix}. $$

Recall that the Gram-Schmidt algorithm turns $A$ into an orthogonal matrix by the following sequence of steps. First normalise $A_1$ (i.e. make it unit length):

$$A_1 \rightsquigarrow \frac{A_1}{|A_1|},$$

or equivalently multiply $A$ on the right by

$$\begin{pmatrix} 1/|A_1| & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$
Next make $A_2$ orthogonal to $A_1$

$$A_2 \leadsto A_2 - \langle A_1, A_2 \rangle A_1,$$

or equivalently multiply by

$$\begin{pmatrix}
1 & -\langle A_1, A_2 \rangle \\
1 & 1 \\
1 & \cdots
\end{pmatrix},$$

and normalise $A_2$:

$$A_2 \leadsto A_2 \frac{1}{|A_2|} \begin{pmatrix}
1 \\
1/|A_2| \\
1 \\
\vdots
\end{pmatrix}.$$

Next make $A_3$ orthogonal to $A_1$ and $A_2$

$$A_3 \leadsto A_3 - \langle A_1, A_3 \rangle A_1 - \langle A_2, A_3 \rangle A_2,$$

and normalise it

$$A_3 \leadsto A_3 \frac{1}{|A_3|} \begin{pmatrix}
1 \\
1/|A_3| \\
1 \\
\vdots
\end{pmatrix}.$$

Continue in this way through $A_n$.

Let $T \subset GL_n \mathbb{R}$ be the subgroup of upper triangular matrices with positive entries along the diagonal, and define $f : O(n) \times T \to GL_n \mathbb{R}$ by $f(A, B) = AB$. This is continuous, although not a group homomorphism. From the discussion above we see that the Gram-Schmidt algorithm is a continuous map $g : GL_n \mathbb{R} \to O(n) \times T$ with $f \circ g = \text{id}$, so $f$ is surjective and $g$ is injective. But $f$ is also injective, as follows. Suppose we have $A, A' \in O(n)$ and $B, B' \in T$ with $f(A, B) = f(A', B')$. Then $AB = A'B'$, so

$$(A')^{-1}A = B'B^{-1}.$$  

The left-hand side is in $O(n)$ and the right-hand side in $T$, and $O(n) \cap T = \{1\}$, so both sides are 1, so $A = A'$ and $B = B'$, as promised. Thus $f$ is a homeomorphism.

The inclusion of the identity matrix (thought of as a point) in $T \cong \mathbb{R}^{n(n+1)/2}$ is a homotopy equivalence, so the inclusion $O(n) \times \{1\} \hookrightarrow O(n) \times T \cong GL_n$ is a homotopy equivalence.

2. Let $G$ be a path-connected, locally path-connected topological group and $p : \tilde{G} \to G$ a path-connected covering space, and fix a point $\bar{1} \in p^{-1}(1) \subset \tilde{G}$.

(a) Show that there is a unique map $\tilde{m} : \tilde{G} \times \tilde{G} \to \tilde{G}$ such that $\tilde{m}(\bar{1}, \bar{1}) = \bar{1}$ and

$$p(\tilde{m}(g, h)) = m(p(g), p(h))$$

for all $g, h \in \tilde{G}$. Hint: Use Proposition 1.33.
Solution: Consider the diagram

\[
\begin{array}{ccc}
\tilde{G} \times \tilde{G} & \overset{\tilde{m}}{\longrightarrow} & \tilde{G} \\
p \times p \downarrow & & \downarrow p \\
G \times G & \overset{m \circ (p \times p)}{\longrightarrow} & G.
\end{array}
\]

To use Proposition 1.33 we must show that im\((m \circ (p \times p))\) \(\subset\) im\(p_*\) in \(\pi_1(G)\). Let \(\gamma_1, \gamma_2 \in \pi_1(\tilde{G})\), so \(m_*(p_*(\gamma_1), p_*(\gamma_2))\) is a typical element of im\((m \circ (p \times p))\). We take \(1 \in G\) as our basepoint, so the identity in \(\pi_1(G)\) is represented by the constant loop 1, so \(m_*\) satisfies \(m_*(\gamma, 1) = m_*(1, \gamma) = \gamma\). By problem 5 last time this gives us \(m_*(p_*(\gamma_1), p_*(\gamma_2)) = p_*(\gamma_1)p_*(\gamma_2)\), which equals \(p_*(\gamma_1\gamma_2)\) and thus is in im\(p_*\). Thus there is a lift \(\tilde{m}\) as in the diagram, sending \((\tilde{1}, \tilde{1})\) to \(\tilde{1}\). By Proposition 1.34 this lift is unique.

(b) Show that the map \(\tilde{m}\) constructed in part (a) satisfies the first two axioms above, with \(\tilde{1}\) in place of 1. Hint: Use Proposition 1.34.

Solution: For the first axiom (associativity), consider the diagrams

\[
\begin{array}{ccc}
(g, h, k) & \overset{\tilde{m}((\tilde{m}(g, h), k))}{\longrightarrow} & (g, h, k) \\
p \times p \times p \downarrow & & \downarrow p \\
G \times G \times G & \overset{m((m(g, h), k))}{\longrightarrow} & G
\end{array}
\]

Because \(G\) is associative, the bottom maps are the same, so the diagonal maps are the same. Thus the top maps lift the same map and agree at \(\tilde{1}\) — they both send it to \(\tilde{1}\) — so by Proposition 1.34 they agree on all of \(\tilde{G} \times \tilde{G} \times \tilde{G}\). For the second axiom (identity), consider the diagrams

\[
\begin{array}{ccc}
g & \overset{\tilde{m}(\tilde{1}, g)}{\longrightarrow} & \tilde{G} \\
p \downarrow & & \downarrow p \\
G & \overset{m(1, g)}{\longrightarrow} & G
\end{array}
\]

The argument is entirely similar: the bottom maps are the same, so the diagonal maps are the same, so the top maps lift the same map and agree at \(\tilde{1}\), so they agree on all of \(\tilde{G}\).

• By a similar argument (which you need not give), there is a unique \(\tilde{i} : \tilde{G} \rightarrow \tilde{G}\) that satisfies the third axiom above. Thus once you have chosen \(\tilde{1}\), there is a unique group structure on \(\tilde{G}\) that makes \(p\) into a group homomorphism. Observe that the kernel of \(p\) (as a group homomorphism) is the fibre \(p^{-1}(1)\).

(c) Let \(K\) be a discrete, normal subgroup of \(\tilde{G}\). Show that \(K\) lies in the centre \(Z(\tilde{G})\). Conclude that \(\ker p \leq Z(\tilde{G})\).
Solution: Let $k \in K$ and $g \in \tilde{G}$. We wish to show that $gkg^{-1} = k$. Since $\tilde{G}$ is path-connected, choose a path $\gamma$ in $\tilde{G}$ from $\tilde{1}$ to $g$. Since $K$ is normal, $\gamma(t) \cdot k \cdot \gamma(t)^{-1}$ is a path in $K$ from $k$ to $gkg^{-1}$. Since $K$ is discrete, any path in $K$ is constant, so $k = gkg^{-1}$.

Now the kernel of a homomorphism is a normal subgroup, and the fibre of a covering space is discrete, so $\ker p$ is a discrete normal subgroup of $\tilde{G}$, hence is contained in $Z(\tilde{G})$.

- From our general theory we know that if $\tilde{G}$ is simply connected then $p^{-1}(1) \cong \pi_1(G,1)$ as sets. It is also true as groups. Think about this.

Thought: Let $\varphi : p^{-1}(1) \to \pi_1(G)$ be the usual bijection; we will show that it is a group homomorphism. Recall that $\varphi$ is defined as follows: given a point $k \in p^{-1}(1)$, choose a path $\gamma$ in $\tilde{G}$ from $\tilde{1}$ to $k$; then $p \circ \gamma$ is a path in $G$ from $p(\tilde{1}) = 1$ to $p(k) = 1$, so let $\varphi(k) = [p \circ \gamma] \in \pi_1(G)$. We have seen that this is well-defined and bijective. To see that it is a homomorphism, let $k_1, k_2 \in \ker p$ and choose paths $\gamma_1$ and $\gamma_2$ in $\tilde{G}$ from $\tilde{1}$ to $k_1$ and $k_2$ respectively. Then $\tilde{m}(\gamma_1(t), \gamma_2(t))$ is a path from $\tilde{1}$ to $\tilde{m}(k_1, k_2)$, and $p(\tilde{m}(\gamma_1(t), \gamma_2(t))) = m(p(\gamma_1(t)), p(\gamma_2(t)))$, so $\varphi(k_1 k_2) = m_*(\varphi(k_1), \varphi(k_2)) = \varphi(k_1) \varphi(k_2)$.

If $\tilde{G}$ is not simply connected, the same argument shows that $\ker p = \pi_1(G)/p_*(\pi_1(\tilde{G}))$. Note that since $\pi_1(G)$ is Abelian, every subgroup is normal.

3. Let $G$ be a finite group. Show that any homomorphism $\varphi : G \to \mathbb{Z}$ is zero.

Solution: Let $g \in G$ and let $n$ be the order of $g$, which is finite. Then

$$n \varphi(g) = \varphi(g^n) = \varphi(1) = 0,$$

so $\varphi(g) = 0/n = 0$. 

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