Enhanced Coursework

Due May 14, 2011

1. (a) Draw pictures of $S^2 \vee S^1$ and its universal cover.
   (b) Let $X$ be $S^2$ union a line segment connecting the north and south poles. Draw pictures of $X$ and its universal cover.

2. Let $G$ be a finite group. Show that any homomorphism $\varphi : G \to \mathbb{Z}$ is zero.

3. The Borsuk–Ulam theorem states that for any map $f : S^2 \to \mathbb{R}^2$ there is an $x \in S^2$ with $f(-x) = f(x)$. The proof reduces to the following lemma: there is no odd map $g : S^2 \to S^1$, i.e. no map with $g(-x) = -g(x)$ for all $x \in S^2$. In lecture we proved this lemma by showing that an odd map $S^1 \to S^1$ induces a nonzero map on $\pi_1$, hence is not nullhomotopic, and considered $g$ restricted to the equator to get a contradiction. Give a second proof of the lemma, as follows.
   (a) Let $q : S^2 \to \mathbb{RP}^2$ be the universal cover and $p : S^1 \to S^1$ the double cover $p(z) = z^2$. Observe that an odd map $g : S^2 \to S^1$ descends to a map $h$ as in this diagram:

   \[
   \begin{array}{ccc}
   S^2 & \xrightarrow{g} & S^1 \\
   q \downarrow & & \downarrow p \\
   \mathbb{RP}^2 & \xrightarrow{h} & S^1.
   \end{array}
   \]

   (b) Use the lifting criterion to produce a map $\tilde{h} : \mathbb{RP}^2 \to S^1$ lifting $h$.

   \[
   \begin{array}{ccc}
   S^1 & \xrightarrow{p} & S^1 \\
   \tilde{h} \downarrow & & \downarrow h \\
   \mathbb{RP}^2 & \xrightarrow{h} & S^1.
   \end{array}
   \]

   (c) Use the unique lifting property to show that $\tilde{h} \circ q = g$, which contradicts the assumption that $g$ was odd.

4. Recall that $\mathbf{Toph}$, is the category whose objects are pointed spaces and whose arrows are homotopy classes of maps (homotopic rel. basepoint), and $\mathbf{Grp}$ is the category whose objects are groups and arrows are group homomorphisms.
   A map $f : (X, x_0) \to (S^1, 1)$ induces a map $f_* : \pi_1(X, x_0) \to \pi_1(S^1, 1) = \mathbb{Z}$. 

1
Show that the map
\[ \Hom_{\text{Top}_*}((X,x_0),(S^1,1)) \to \Hom_{\text{Grp}}(\pi_1(X,x_0),\mathbb{Z}) \]
\[ f \mapsto f_* \] (\ast)

is injective, as follows.

(a) Let \( G \) be any group, \( e \in G \) the identity, and \( \varphi : G \times G \to G \) a homomorphism with \( \varphi(g,e) = g = \varphi(e,g) \) for all \( g \in G \). [At this point I meant to write “Conclude that \( \varphi(g,h) = gh \) for all \( g,h \in G \).” Many of you figured this out.] (Optional: Conclude that \( G \) is abelian.)

(b) Let \( m : S^1 \times S^1 \to S^1 \) be the multiplication map \( m(z,w) = zw \).
Apply part (a) to \( m_* : \pi_1(S^1) \times \pi_1(S^1) \to \pi_1(S^1) \) to conclude that \( m_* \) is the multiplication in \( \pi_1(S^1) \). (We are dealing with several groups here: \( S^1 \) with complex multiplication, \( \pi_1(S^1,1) \) with concatenation of paths, and \( \mathbb{Z} \) with addition. Do not get confused about which group you are in.)

(c) Define a group structure on the left-hand side of (\ast). (Use \( m \) to multiply elements of \( S^1 \) or we’ll all go nuts.)

(d) Define a group structure on the right-hand side of (\ast).

(e) Show that the map in (\ast) is a group homomorphism.

(f) Let \((X,x_0)\) be any pointed space and \( f : (X,x_0) \to (S^1,1) \) a map such that \( f_* : \pi_1(X,x_0) \to \mathbb{Z} \) is the zero map. Show that \( f \) is nullhomotopic. (Hint: Consider the universal cover \( p : \tilde{X} \to S^1 \).)

Conclude that the map in (\ast) is injective.

5. Optional: * Suppose that \( X \) is good in the sense that it has a universal cover and admits partitions of unity. Show that the map in (\ast) above is also surjective, as follows. Let \( G = \pi_1(X,x_0) \) and \( p : \tilde{X} \to X \) be the universal cover. Recall that \( \tilde{X} \) looks locally like \( X \times G \); that is, we can choose an open cover \( \{U_\alpha\} \) of \( X \) and maps \( \psi_\alpha : p^{-1}(U_\alpha) \to G \), each of which gives a bijection with between the fibres of \( p \) and \( G \). Choose a partition of unity \( \{\rho_\alpha\} \) subordinate to the cover \( \{U_\alpha\} \).

Now given a homomorphism \( \varphi : G \to \mathbb{Z} \), define a map \( F : \tilde{X} \to \mathbb{R} \) by
\[ F(\tilde{x}) = \sum_\alpha \rho_\alpha(p(\tilde{x})) \cdot \varphi(\psi_\alpha(\tilde{x})) \]

Observe that this is well-defined. Show that it descends to a map \( f : X \to S^1 \) with \( f_* = \varphi \).

*Really. You have better things to do with your time than work on this problem.