Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.
All “spaces” are path connected, locally path connected Hausdorff topological spaces, and all “maps” are continuous.

1. Let $f : \mathbb{C} \to \mathbb{C}$ be the function $f(z) = z^2$.
   (a) Let $g$ be the restriction of $f$ to the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus 0$. Describe $g_* : \pi_1(\mathbb{C}^*) \to \pi_1(\mathbb{C}^*)$. (No proofs required.)
   (b) Show that there is no continuous “square root function” $h : \mathbb{C} \to \mathbb{C}$ such that $f(h(z)) = z$ for all $z \in \mathbb{C}$.
   (c) Suppose that $\phi : \mathbb{C} \to \mathbb{C}$ satisfies $\phi|_{S^1} = f|_{S^1}$. Show that there is a $z \in \mathbb{C}$ with $\phi(z) = 0$.

2. (a) State van Kampen’s theorem.
   (b) An $n$-manifold is a space in which every point has a neighbourhood homeomorphic to $\mathbb{R}^n$. Show that if $X$ is a path-connected $n$-manifold with $n \geq 3$ and $x \in X$ is any point then $\pi_1(X) \cong \pi_1(X \setminus \{x\})$.
   (c) Give an example of a 2-manifold (i.e a surface) $X$ where $\pi_1(X) \not\cong \pi_1(X \setminus \{x\})$ for some $x \in X$, and an example where $\pi_1(X) \cong \pi_1(X \setminus \{x\})$.

3. (a) Write down the long exact sequence for the reduced homology groups of a pair $(X, A)$, giving a very brief description of the maps involved. (No need to check these maps are well-defined, or the sequence is exact. Sufficient to define them briefly on chains.)
   We define $CX := X \times [0, 1]/(X \times \{0\})$, and $SX := CX/(X \times \{1\})$. Throughout this question you may use any results without proof that you state correctly.
   (b) What are the reduced homology groups of $CX$?
   (c) What are the reduced homology groups of $SX$?
   (d) What are $C(S^n)$ and $S(S^n)$? (No proof necessary.)
   Deduce the reduced homology groups of $S^n$. 

4. Let \( p : (\tilde{X}, \tilde{x}_0) \to (X, x_0) \) be a cover and \( f : (Y, y_0) \to (X, x_0) \) be any map.

(a) State a necessary and sufficient condition for the existence of a map \( \tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0) \) with \( p \circ \tilde{f} = f \).

(b) Prove that this condition is necessary.

(c) State the unique lifting property.

(d) If \( Y \) is simply connected, show that the map

\[
\text{Hom}_{\text{Top}}(((Y, y_0), (\tilde{X}, \tilde{x}_0)) \to \text{Hom}_{\text{Top}}(((Y, y_0), (X, x_0)), \quad \tilde{f} \mapsto p \circ \tilde{f}
\]

is injective and surjective.

(e) (The universal cover is a functor.) Suppose that \( \tilde{X} \) is simply connected (so \( p \) is the universal cover of \( X \)) and \( q : (\tilde{Y}, \tilde{y}_0) \to (Y, y_0) \) is the universal cover of \( Y \). Show that any map \( f : (Y, y_0) \to (X, x_0) \) induces a map \( \tilde{F} : (\tilde{Y}, \tilde{y}_0) \to (\tilde{X}, \tilde{x}_0) \) making a commutative diagram:

\[
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\tilde{F}} & \tilde{X} \\
\downarrow{q} & & \downarrow{p} \\
Y & \xrightarrow{f} & X.
\end{array}
\]
Answers

1. (a) \( g_\ast \) is the map \( \mathbb{Z} \to \mathbb{Z} \) given by \( n \mapsto 2n \). (seen, 4 marks)

(b) Suppose there were such an \( h \) and let \( k = h|_{\mathbb{C}^\ast} \). We can view \( k \) as a map \( \mathbb{C}^\ast \to \mathbb{C}^\ast \), for if \( h(z) = 0 \) then \( z = f(h(z)) = f(0) = 0 \). Now the composition

\[
\mathbb{C}^\ast \xrightarrow{k} \mathbb{C}^\ast \xrightarrow{g} \mathbb{C}^\ast
\]

is the identity, so looking at the induced maps on \( \pi_1 \)

\[
\mathbb{Z} \xrightarrow{k_\ast} \mathbb{Z} \xrightarrow{g_\ast} \mathbb{Z}
\]

we have \( g_\ast \circ k_\ast = 1_{\mathbb{Z}} \). But this would imply that \( 1 = g_\ast(k_\ast(1)) = 2k_\ast(1) \), which is impossible in \( \mathbb{Z} \). (seen, 8 marks)

(c) If there were no such \( z \), we could view \( \phi \) as a map \( \mathbb{C} \to \mathbb{C}^\ast \). Consider

\[
S^1 \xrightarrow{i} \mathbb{C} \xrightarrow{\phi} \mathbb{C}^\ast,
\]

where \( i \) is the inclusion. Looking at the induced maps on \( \pi_1 \) we get

\[
\mathbb{Z} \xrightarrow{i_\ast} 0 \xrightarrow{\phi_\ast} \mathbb{Z},
\]

so \( \phi_\ast \circ i_\ast = 0 \). But this contradicts the fact that \( \phi \circ i = f|_{S^1} \) and \( (f|_{S^1})_\ast \) is multiplication by 2. (seen similar, 8 marks)

2. (a) Suppose that \( X = U \cup V \), where \( U \) and \( V \) are open and path-connected and \( U \cap V \) is path-connected. Fix a basepoint \( p \in U \cap V \). Label the inclusions as

\[
\begin{array}{ccc}
U \cap V & \xrightarrow{k} & U \\
\downarrow i & & \downarrow i \\
V & \xrightarrow{j} & X.
\end{array}
\]

Then the map \( i_\ast \circ j_\ast : \pi_1(U,p) \ast \pi_1(V,p) \to \pi_1(X,p) \) is surjective and its kernel is the normal subgroup \( N \) generated by

\[
\{k_\ast(\gamma) \ast l_\ast(\gamma)^{-1} : \gamma \in \pi_1(U \cap V,p)\}.
\]

That is, it induces an isomorphism \( \pi_1(X,p) \cong (\pi_1(U,p) \ast \pi_1(V,p))/N \). Also ok to say

\[
\pi_1(X,p) \cong \frac{\pi_1(U,p) \ast \pi_1(V,p)}{k_\ast(\gamma) = l_\ast(\gamma) \text{ for all } \gamma \in \pi_1(U \cap V,p)}.
\]

(see, 4 marks)

(b) Let \( U = X \setminus x \) and \( V \) be a neighbourhood of \( x \) homeomorphic to \( \mathbb{R}^n \). Then \( U \), \( V \), and \( U \cap V \) are path-connected, \( \pi_1(V) = 1 \), and \( U \cap V = V \setminus x \) is homotopy equivalent to \( S^{n-1} \), so \( \pi_1(U \cap V) = 1 \) as well. Thus van Kampen’s theorem states that \( i_\ast : \pi_1(U) \to \pi_1(X) \) is an isomorphism. (unseen, 6 marks)
3. (a) Let $i : A \to X$ be the inclusion. Then the long exact sequence is

$$\ldots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{\pi_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \to \ldots$$

Here $i_*$ takes a simplex (singular or simplicial) $\sigma : \Delta^n \to A$ to $i \circ \sigma : \Delta^n \to X$ and extends by linearity to map chains in $A$ to chains in $X$ (and restricts to map cycles in $A$ to cycles in $X$).

(b) $\partial_*$ maps chains (or cycles) on $X$ (i.e. elements of $C_n(X)$) to their equivalence class in $C_n(X, A) := C_n(X)/C_n(A)$.

(c) The condition is that $\pi_1(X \setminus x) = \pi_1(D) \cong \pi_1(X \setminus \{x\})$.

If instead we take $X = S^2$ then $\{1\} \cong \pi_1(X) \cong \pi_1(D^2) \cong \pi_1(X \setminus \{x\})$.

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$\pi_*$ maps chains (or cycles) on $X$ (i.e. elements of $C_n(X)$) to their equivalence class in $C_n(X, A) := C_n(X)/C_n(A)$.

$\partial_*$ takes a cycle $\sigma$ in $C_n(X)/C_n(A)$, lifts it to a chain $\tilde{\sigma}$ in $C_n(X)$ then notes that its boundary $\partial \tilde{\sigma}$ is in the image of $i_*$ (since $\pi(\partial \tilde{\sigma}) = \partial \sigma = 0$), $\partial \tilde{\sigma} = i_*(a)$. We set $\partial \sigma := [a]$.

(b) $CX$ deformation retracts to its basepoint (the image of $X \times \{0\}$) via the retraction $f_1 : CX \to CX$, $f_1((x, s)) := (x, st)$. So $CX \sim \{\text{point}\}$, which has $\tilde{H}_n = 0$. Since homology is a homotopy invariant, $\tilde{H}_n(CX) = 0 \forall n$.

(c) Since $X \times \{1\}$ has an open neighbourhood $X \times (0.9, 1) \subset CX$ of which it is a deformation retract, $\tilde{H}_n(CX, X \times \{1\}) \cong \tilde{H}_n(CX/(X \times \{1\})) = \tilde{H}_n(SX)$.

Combining this with $\tilde{H}_n(CX) = 0 \forall n$, the long exact sequence of the pair $(CX, X \times \{1\})$ becomes

$$\ldots \to 0 \to \tilde{H}_n(SX) \xrightarrow{\partial_*} \tilde{H}_{n-1}(X) \to 0 \to \ldots$$

By exactness, then, $\tilde{H}_n(SX) \cong \tilde{H}_{n-1}(X)$.

(d) $C(S^n) \cong D^{n+1}$, $S(S^n) \cong S^{n+1}$.

Inductively, we conclude from (c) that $\tilde{H}_k(S^n) \cong \tilde{H}_{k-n}(S^0)$, which is $\mathbb{Z}$ if $k = n$ and 0 otherwise.

4. (a) The condition is that $f_* (\pi_1(Y, y_0)) \subset \pi_1(\tilde{X}, \tilde{x}_0))$.

(b) If $f = p \circ \tilde{f}$ then $f_* = p_* \circ \tilde{f}_*$, so

$$f_* (\pi_1(Y, y_0)) = p_* (\tilde{f}_*(\pi_1(Y, y_0))) \subset \pi_1(\tilde{X}, \tilde{x}_0))$$

since $\tilde{f}_*(\pi_1(Y, y_0)) \subset \pi_1(\tilde{X}, \tilde{x}_0))$.

(c) Two maps $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ with $p \circ \tilde{f}_1 = p \circ \tilde{f}_2$ are equal if $\tilde{f}_1(y) = \tilde{f}_2(y)$ for some $y \in Y$ (ok if replace some $y$ with the basepoint $y_0$).

(d) If we have two maps $\tilde{f}_1, \tilde{f}_2 : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ then $\tilde{f}_1(y_0) = \tilde{x}_0 = \tilde{f}_2(y_0)$, so if $p \circ \tilde{f}_1 = p \circ \tilde{f}_2$ then $\tilde{f}_1 = \tilde{f}_2$ by the unique lifting property, so the map is injective. If $f : (Y, y_0) \to (X, x_0)$ then $f_* (\pi_1(Y, y_0)) = f_* (1) = 1 \subset p_* (\pi_1(\tilde{X}, \tilde{x}_0))$, so by the lifting criterion, there is an $\tilde{f}$ with $p \circ \tilde{f} = f$, so the map is surjective.

(e) Since $\tilde{Y}$ is simply connected, we can apply the lifting criterion to $f \circ q$ to get $\tilde{F}$. (unseen, but they've seen functors 5 marks)