Solutions to Homework 2

1. Let $X$ and $Y$ be topological spaces. Show that homotopy is an equivalence relation on $\text{Hom}(X, Y)$.

**Solution:** First we show that homotopy is reflexive. Given $f : X \to Y$, the map $h : X \times I \to Y$ given by $(x, t) \mapsto f(x)$ is a homotopy from $f$ to $f$.

Next we show that homotopy is symmetric. Given $f, g : X \to Y$ and a homotopy $h$ from $f$ to $g$, the map $X \times I \to Y$ given by $(x, t) \mapsto h(x, 1-t)$ is a homotopy from $g$ to $f$.

Last we show that homotopy is transitive. Given $f, g, k : X \to Y$, a homotopy $h$ from $f$ to $g$, and a homotopy $h'$ from $g$ to $k$, define a map $H : X \times I \to Y$ by

$$H(x, t) = \begin{cases} h(x, 2t) & \text{if } t \in [0, \frac{1}{2}] \\ h'(x, 2t-1) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

This is well-defined when $t = \frac{1}{2}$ because $h(x, 1) = g(x) = h'(x, 0)$. It is continuous on the closed subsets $X \times [0, \frac{1}{2}]$ and $X \times [\frac{1}{2}, 1]$ of $X \times I$, hence is continuous on the whole domain. Lastly $H(x, 0) = h(x, 0) = f(x)$ and $H(x, 1) = h'(x, 1) = k(x)$, so $H$ is a homotopy from $f$ to $k$.

2. Let $X, Y$, and $Z$ be topological spaces, and $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$. If $f_0 \simeq f_1$ and $g_0 \simeq g_1$, show that $g_0 \circ f_0 \simeq g_1 \circ f_1$.

**Solution:** Let $h : X \times I \to Y$ be a homotopy from $f_0$ to $f_1$ and $h' : Y \times I \to Z$ a homotopy from $g_0$ to $g_1$. Define $H : X \times I \to Z$ by $H(x, t) = h'(h(x, t), t)$. Then $H$ is continuous, $H(x, 0) = h'(h(x, 0), 0) = g_0(f_0(x))$, and $H(x, 1) = h'(h(x, 1)) = h_1(f_1(x))$, so $H$ is a homotopy from $g_0 \circ f_0$ to $g_1 \circ f_1$.

3. Let $X$ be a topological space. Show that the following are equivalent:

   (a) $X$ is contractible.

   (b) For every space $Y$, every map $f : X \to Y$ is nullhomotopic.

   (c) For every space $Z$, every map $g : Z \to X$ is nullhomotopic.

**Solution:** Following the hint, we will use the fact that $X$ is contractible if and only if the identity map $1 : X \to X$ is nullhomotopic.

$(a) \Rightarrow (b)$ and $(c)$. Since $X$ is contractible, the identity map $1 : X \to X$ is homotopic to a constant map $c : X \to X$. Thus $f = f \circ 1 \simeq f \circ c$, which is a constant map, and similarly $g = 1 \circ g \simeq c \circ g$, which is constant.

$(b) \Rightarrow (a)$. Take $Y = X$ and $f = 1$. Then $1$ is nullhomotopic, so $X$ is contractible.
(c) ⇒ (a): Take \( Z = X \) and \( g = 1 \). Then \( 1 \) is nullhomotopic, so \( X \) is contractible.

4. Let \( X \) be a topological space. Define \( x \sim x' \) if there is a path from \( x \) to \( x' \). We define \( \pi_0(X) = X/\sim \). 

(a) Show that a map \( f : X \to Y \) induces a map \( \pi_0(X) \to \pi_0(Y) \).

Solution: If \( x \in X \), let \([x] \in \pi_0(X)\) denote the path component of \( X \). Define \( f_* : \pi_0(X) \to \pi_0(Y) \) by \( f_*([x]) = [f(x)] \). This is well-defined, as follows: if \( x \sim x' \), let \( \gamma : I \to X \) be a path from \( x \) to \( x' \); then \( f \circ \gamma \) is a path from \( f(x) \) to \( f(x') \), so \( f(x) \sim f(x') \).

(b) Show that homotopic maps \( X \to Y \) induce the same map on \( \pi_0 \).

Solution: Let \( f, g : X \to Y \) be homotopic maps and \( h : X \times I \to Y \) a homotopy from \( f \) to \( g \). Given \( x \in X \), let \( \gamma(t) = h(x, t) \); then \( \gamma \) is a path from \( f(x) \) to \( g(x) \), so \( f(x) \sim g(x) \). Thus \( f_*([x]) = [f(x)] = [g(x)] = g_*([x]) \).

(c) Show that a homotopy equivalence induces a bijection on \( \pi_0 \).

Solution: First observe that if \( f : X \to Y \) and \( g : Y \to Z \) then for every \( x \in X \) we have \( g_* (f_* ([x])) = g_* ([f(x)]) = [g(f(x))] = \) \( (g \circ f)_* ([x]) \); that is, \( g_* \circ f_* = (g \circ f)_* \). Next observe that if \( 1 : X \to X \) is the identity then \( 1_* : \pi_0(X) \to \pi_0(X) \) is also the identity.

Now let \( f : X \to Y \) be a homotopy equivalence with homotopy inverse \( g : X \to Y \). Then \( g \circ f \simeq 1 \), so \( g_* \circ f_* = (g \circ f)_* = 1_* = 1 \). Similarly, \( f \circ g \simeq 1 \), so \( f_* \circ g_* = 1 \). Thus \( g_* \) is an inverse to \( f_* \), so \( f_* \) is a bijection.

5. Let \( X \) be a topological space and \( f : S^1 \to X \). Show that the following are equivalent:

(a) \( f \) is nullhomotopic.

(b) \( f \) extends to a map \( D^2 \to X \); that is, there is a map \( g : D^2 \to X \) such that the restriction \( g|_{\partial D^2} = f \).

Solution: By definition, a homotopy from a constant map to \( f \) is a map \( h : S^1 \times I \to X \) that is constant on \( S^1 \times \{0\} \) and whose restriction to \( S^1 \times \{1\} \) is \( f \). In problem 4 last week we exhibited a homeomorphism \( (S^1 \times I)/(S^1 \times \{0\}) \to D^2 \) which mapped \( S^1 \times \{1\} \) to \( \partial D^2 \); thus maps \( h \) as above are in bijection with maps \( g : D^2 \to X \) whose restriction to \( \partial D^2 \) is \( f \).