Homework 4

Due Friday, February 11, 2011

To summarise what I said on Thursday: a score of 3/4 or 2/4 does not mean that a quarter or half of your work is rubbish, but that you made one or two mistakes. I appreciate that everyone is working very hard on these homeworks, and I have been very pleased with the quality of the work. If the totals at the end of the term don't reflect that, I will rescale them so that they do (uniformly for the whole class, of course).

1. (a) Give an example of a map $f : (X, p) \to (Y, q)$ which is injective but not surjective, for which the induced map $f_* : \pi_1(X, p) \to \pi_1(Y, q)$ is injective but not surjective. (Don't prove anything, just say what f and f_* are.)
   (b) Give an example where f is injective but not surjective and f_* is surjective but not injective.
   (c) Give an example where f is surjective but not injective and f_* is injective but not surjective.
   (d) Give an example where f is surjective but not injective and f_* is surjective but not injective.
   (e) Give any other examples along these lines that you think are interesting. (You can leave this blank.)

2. Let G be a group.
   (a) Say what it means for a subgroup N of G to be *normal*.
   (b) (Universal property of the quotient group.) If N is a normal subgroup of G, the natural map $\rho : G \to G/N$ kills N, that is, $\rho(N) = 1$. Show that any other such map factors through ρ: that is, if a map* $\varphi : G \to H$ kills N then there is a unique map $\psi : G/N \to H$ with $\psi \circ \rho = \varphi$.

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[dr, "\varphi"'] & G/N \arrow[d, dotted, "\psi"] \\ & H \end{tikzcd}$$

*This means a homomorphism. We wouldn't talk about a map between groups that wasn't a homomorphism.

3. Let G be a group. The *commutator* of two elements $g, h \in G$ is $[g, h] = ghg^{-1}h^{-1}$ (so called because g and h commute if and only if their commutator is 1). The *commutator subgroup* of G is

$$[G, G] = \langle [g, h] : g, h \in G \rangle.$$

Here the angle brackets mean "the subgroup generated by..."; a general element of $[G, G]$ is not a single commutator but a product of commutators.

(a) Show that $[G, G]$ is a characteristic subgroup of G: that is, for any automorphism σ of G, we have $\sigma([G, G]) = [G, G]$. Conclude that $[G, G]$ is normal.

(b) Show that the quotient group $G/[G, G]$ is Abelian. It is called the *Abelianisation* of G.

(c) (Universal property of the Abelianisation.) The natural map $\rho : G \to G/[G, G]$ is a map from G to an Abelian group. Show that any other map with this property factors through ρ: that is, if A is Abelian and $\varphi : G \to A$ is any map, then there is a unique $\psi : G/[G, G] \to A$ with $\psi \circ \rho = \varphi$.

(d) Consider the dihedral group

$$D_n = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle.$$

(Some authors call this group D_{2n} since its order is $2n$, but they don't call the symmetric group $S_{n!}$, do they.) Describe its Abelianisation. Hint: the answer depends on whether n is odd or even.