Solutions to Homework 4

1. (a) Give an example of a map \( f : (X,p) \to (Y,q) \) which is injective but not surjective, for which the induced map \( f_* : \pi_1(X,p) \to \pi_1(Y,q) \) is injective but not surjective.

**Solution:** Let \( f \) be the inclusion of the circle \( S^1 \times \{1\} \) in the torus \( S^1 \times S^1 \). Then \( f_* \) is the inclusion \( Z \times 0 \to Z \times Z \). Or let \( f : [0,\frac{1}{2}] \to S^1 \) be defined by \( f(x) = e^{2\pi ix} \), so \( f_* : 0 \to Z \).

(b) Give an example where \( f \) is injective but not surjective and \( f_* \) is injective but not surjective.

**Solution:** Let \( f \) be the inclusion of the torus \( S^1 \times S^1 \) in the solid torus \( S^1 \times D^2 \). Then \( f_* : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \) is projection onto the first factor. Or let \( f \) be the inclusion of \( S^1 \) in \( S^2 \) as the equator, so \( f_* : \mathbb{Z} \to 0 \).

(c) Give an example where \( f \) is surjective but not injective and \( f_* \) is injective but not surjective.

**Solution:** Let \( f : S^1 \to S^1 \) be defined by \( f(z) = z^2 \). Then \( f_* : \mathbb{Z} \to \mathbb{Z} \) is multiplication by 2. Or let \( f : [0,1] \to S^1 \) be defined by \( f(x) = e^{2\pi ix} \), so \( f_* : 0 \to \mathbb{Z} \).

(d) Give an example where \( f \) is surjective but not injective and \( f_* \) is surjective but not injective.

**Solution:** Let \( f : S^1 \times S^1 \to S^1 \) be projection onto the first factor. Then \( f_* : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \) is projection onto the first factor. Or let \( f : S^1 \times I \to D^2 \) be defined by \( f(z,t) = tz \), so \( f_* : \mathbb{Z} \to 0 \).

(e) Give any other examples along these lines that you think are interesting.

**Solution:** Consider the inclusion \( i : S^1 \to \mathbb{C} \setminus \{0\} \) and the retraction \( r \) the other way defined by \( r(z) = z/|z| \). Then \( i \) is injective but not surjective, \( r \) is surjective but not injective, and \( i_* \) and \( r_* \) are both bijective.

2. Let \( G \) be a group.

(a) Say what it means for a subgroup \( N \) of \( G \) to be normal.

**Solution:** For every \( n \in N \) and \( g \in G \) we have \( gng^{-1} \in N \). (Note that we do not require \( gng^{-1} = n \); that is a central subgroup, not a normal subgroup.)

(b) Show that if \( \varphi : G \to H \) kills \( N \) then there is a unique map \( \psi : G/N \to H \) with \( \psi \circ \rho = \varphi \), where \( \rho : G \to G/N \) is the natural map.

**Solution:** Given an element \( x \in G/N \), choose a \( g \in G \) with \( \rho(g) = x \) and define \( \psi(x) = \varphi(g) \). If \( g' \) is another element with \( \rho(g') = x \) then \( \rho(g'g^{-1}) = xx^{-1} = 1 \), so \( g'g^{-1} \in N \), so \( \varphi(g') = \varphi(g)g^{-1} \varphi(g) = 1 \cdot \varphi(g) \); thus \( \psi \) is well-defined.

Next we show that \( \psi \) is a group map. Given \( x,x' \in G/N \), choose \( g,g' \in G \) with \( \rho(g) = x \) and \( \rho(g') = x' \). Then \( \psi(xx') = \varphi(gg') = \varphi(g)\varphi(g') = \psi(x)\psi(x') \), as desired.

Last we show that \( \psi \) is unique. Suppose that \( \psi' : G/N \to H \) is another map with \( \psi' \circ \rho = \varphi = \psi \circ \rho \). Since \( \rho \) is surjective, this implies that \( \psi' = \psi \), as we saw in the first homework.


**Solution:** Let $\varphi : G \to H$ be any map. For any $g_1, g_2 \in G$ we have

$$\varphi([g_1, g_2]) = \varphi(g_1 g_2 g_1^{-1} g_2^{-1}) = \varphi(g_1)\varphi(g_2)\varphi(g_1)^{-1}\varphi(g_2)^{-1} = [\varphi(g_1), \varphi(g_2)].$$

Thus $\varphi$ takes generators of $[G, G]$ to elements (generators in fact) of $[H, H]$, so $\varphi([G, G]) \subset [H, H]$.

If $\sigma : G \to G$ is an automorphism then $\sigma([G, G]) \subset [G, G]$ and, since $\sigma^{-1}$ is an automorphism, $[G, G] = \sigma([G, G]) \subset \sigma([G, G])$, so $\sigma([G, G]) = [G, G]$, as required.

In particular, for any $g \in G$ the automorphism $\sigma(x) = gxg^{-1}$ preserves $[G, G]$, so $[G, G]$ is normal.

(b) Show that the quotient group $G/[G, G]$ is Abelian.

**Solution:** Let $x, y \in G/[G, G]$. Since the natural map $\rho : G \to G/[G, G]$ is surjective, we can choose $g, h \in G$ with $\rho(g) = x$ and $\rho(h) = y$. Then $[x, y] = \rho([g, h])$, but $[g, h] \in [G, G]$ which $\rho$ kills, so $[x, y] = 1$, so $x$ and $y$ commute.

(c) Show that if $A$ is Abelian and $\varphi : G \to A$ is any map, then there is a unique $\psi : G/[G, G] \to A$ with $\psi \circ \rho = \varphi$.

**Solution:** By problem 2(b), it suffices to show that $\varphi$ kills $[G, G]$, or in fact just the generators of $[G, G]$. If $g, h \in G$ then $\varphi([g, h]) = [\varphi(g), \varphi(h)] = 1$ since $A$ is Abelian.

(d) Describe the Abelianisation of the dihedral group $D_n = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$.

**Solution:** Since $D_n$ is generated by two elements $r$ and $s$, its commutator subgroup is the normal subgroup generated by $[r, s]$. Modding out by this amounts to imposing the relation $[r, s] = 1$, or equivalently $rs = sr$. We already have $rs = sr^{-1}$, so we get

$$1 = [r, s] = rsr^{-1}s^{-1} = r \cdot rs \cdot s^{-1} = r^2.$$

If $n$ is odd then this and $r^n = 1$ imply that $r = 1$, so

$$D_n/[D_n, D_n] = \langle r, s \mid r = s^2 = 1 \rangle = \mathbb{Z}/2.$$

If $n$ is even then

$$D_n/[D_n, D_n] = \langle r, s \mid r^2 = s^2 = 1, rs = sr \rangle = \mathbb{Z}/2 \times \mathbb{Z}/2.$$

2