Homework 4

Due Monday, February 6, 2011

1. Let X be path-connected. Show that the following are equivalent:
   (a) X is simply connected, that is, $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.
   (b) For any two points $x_0, x_1 \in X$, any two paths paths γ, γ' from x_0 to x_1 are homotopic rel. endpoints.

   You may freely use the statements in problem 1(b) last week—we only proved them for paths with $\gamma(0) = \gamma(1)$, but the same proofs work for any path. Just be careful to distinguish between the constant path at x_0 and the constant path at x_1.

2. In the category of sets, every surjection has a section: if X and Y are sets and $f : X \to Y$ a surjection, there is an $s : Y \to X$ with $f \circ s = 1$; this is the axiom of choice. Show that this is not true in the category of topological spaces and continuous maps, as follows.
   (a) Define $\varphi : \mathbb{Z} \to \mathbb{Z}$ by $\varphi(n) = 2n$. Show that there is no homomorphism $\psi : \mathbb{Z} \to \mathbb{Z}$ with $\varphi \circ \psi = 1$.
   (b) Define $f : S^1 \to S^1$ by $f(z) = z^2$. Observe that f is a continuous surjection. Show that there is no continuous $s : S^1 \to S^1$ with $f \circ s = 1$.

3. (a) Give an example of path-connected spaces X and Y and a map $f : X \to Y$ which is injective but not surjective, for which the induced map $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is injective but not surjective for some (and hence every) $x \in X$. Don't prove anything, just say what f and f_* are.
   (b) Give an example where f is injective but not surjective and f_* is surjective but not injective.
   (c) Give an example where f is surjective but not injective and f_* is injective but not surjective.
   (d) Give an example where f is surjective but not injective and f_* is surjective but not injective.
   (e) Give any other examples along these lines that you think are interesting. (You can leave this blank.)

4. What is one question you have about last week's lectures?