Homework 6

Due Monday, February 27, 2011

This homework looks long, but your solutions may well be shorter than the assignment.

1. Consider the following covering of $X = S^1 \vee S^1$:

Let $x_0 \in X$ and $\tilde{x}_0, \tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$ be as in the drawing. Show that $\text{Aut}(\tilde{X} : X) = 1$, as follows.

(a) Let $\gamma$ be the loop based at $x_0$ shown in the diagram. Draw the unique lift $\tilde{\gamma}_0$ of $\gamma$ that starts at $\tilde{x}_0$, and the unique lifts $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ starting at $\tilde{x}_1$ and $\tilde{x}_2$, respectively.

(b) Let $\varphi \in \text{Aut}(\tilde{X} : X)$, that is $\varphi : \tilde{X} \to \tilde{X}$ is a homeomorphism with $p \circ \varphi = p$. If $\varphi(\tilde{x}_0) = \tilde{x}_1$, argue that $\varphi \circ \tilde{\gamma}_0$ is a lift of $\gamma$ that starts at $\tilde{x}_1$, but that $\varphi \circ \tilde{\gamma}_0 \neq \tilde{\gamma}_1$, which is a contradiction.

(c) Write "Similarly, we cannot have $\varphi(\tilde{x}_0) = \tilde{x}_2$, so $\varphi(\tilde{x}_0) = \tilde{x}_0."$

(d) Conclude that $\varphi$ is the identity.

2. Last week we considered the following homotopy equivalent spaces:

(a) The sphere with the north and south poles glued together.

(b) The sphere with a line segment connecting the north and south poles.

(c) The torus with a disc glued along the inner loop.

Draw a simply connected cover of each. Hint: We saw that they had $\pi_1 = \mathbb{Z}$, so the fibre will have to be $\mathbb{Z}$.

(Continued overleaf.)
3. (a) Let \( p : \tilde{X} \to X \) be a covering, \( x_0 \in X \) a basepoint, and \( \tilde{x}_0 \in p^{-1}(x_0) \). Let \( Z \) be a simply-connected space, \( z_0 \in Z \) a basepoint, \( f_0, f_1 : Z \to X \) two maps with \( f_0(z_0) = f_1(z_0) = x_0 \), and \( \tilde{f}_0, \tilde{f}_1 \) the unique lifts with \( \tilde{f}_0(z_0) = \tilde{f}_1(z_0) = \tilde{x}_0 \). Draw a diagram. Show that if \( f_0 \simeq f_1 \text{ rel. basepoint} \) then \( \tilde{f}_0 \simeq \tilde{f}_1 \text{ rel. basepoint} \).

(b) Now suppose that \( p : \tilde{X} \to X \) and \( q : \tilde{Y} \to Y \) are coverings with \( \tilde{X} \) and \( \tilde{Y} \) simply connected. Let \( x_0 \in X \) and \( y_0 \in Y \) be basepoints, \( \tilde{x}_0 \in p^{-1}(x_0) \), and \( \tilde{y}_0 \in q^{-1}(y_0) \). Show that if \( X \simeq Y \text{ rel. basepoint} \) then \( \tilde{X} \simeq \tilde{Y} \text{ rel. basepoint} \).

4. What is one question you have about last week’s lectures?