Solutions to Homework 8

1. Let $X = S^2$, let $A \subset X$ consist of two points, and let $i : A \to X$ be the inclusion.

(a) What is $H_0(A)$? $H_0(X)$? The induced map $i_* : H_0(A) \to H_0(X)$? Its kernel?

**Solution:** By counting path components we see that $H_0(A) = \mathbb{Z}^2$ and $H_0(X) = \mathbb{Z}$. The induced map $i_* : \mathbb{Z}^2 \to \mathbb{Z}$ is $i_*(a, b) = a + b$, by the same argument as for $S^0 \hookrightarrow D^1$ in lecture. Its kernel consists of pairs of the form $(a, -a)$; it is isomorphic to $\mathbb{Z}$.

(b) Compute the homology of $X/A$ using the long exact sequence of the pair.

**Solution:** The long exact sequence is

\[
\cdots \to H_3(A) \xrightarrow{i_*} H_3(X) \to H_3(X, A) \\
\to H_2(A) \xrightarrow{i_*} H_2(X) \to H_2(X, A) \\
\to H_1(A) \xrightarrow{i_*} H_1(X) \to H_1(X, A) \\
\to H_0(A) \xrightarrow{i_*} H_0(X) \to H_0(X, A) \to 0.
\]

We know $H_*(A)$ and $H_*(X)$, so the long exact sequence becomes

\[
\cdots \to 0 \to 0 \to H_3(X, A) \\
\to 0 \to \mathbb{Z} \to H_2(X, A) \\
\to 0 \to 0 \to H_1(X, A) \\
\to \mathbb{Z}^2 \xrightarrow{i_*} \mathbb{Z} \to H_0(X, A) \to 0.
\]

From

\[
0 \to H_3(X, A) \to 0
\]

we see that $H_3(X, A) = 0$, and similarly for $H_4(X, A)$ and higher. From

\[
0 \to \mathbb{Z} \to H_2(X, A) \to 0
\]

we see that $H_2(X, A) = \mathbb{Z}$. From

\[
0 \to H_1(X, A) \to \mathbb{Z}^2 \xrightarrow{i_*} \mathbb{Z}
\]

we see that $H_1(X, A) = \ker i_* = \mathbb{Z}$.
For $n \geq 1$ we have

$$H_n(X/A) = H_n(X/A, \text{point}) = H_n(X, A).$$

Since $X/A$ is path-connected, $H_0(X/A) = \mathbb{Z}$. Thus

$$H_n(X/A) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & n \geq 3. \end{cases}$$

2. Show that the antipodal map $S^n \to S^n$ is homotopic to the identity if $n$ is odd.

**Solution:** If $n = 2m - 1$ then $S^n$ is the unit sphere in $\mathbb{C}^m$:

$$S^n = \{ \vec{z} \in \mathbb{C}^m : |\vec{z}| = 1 \}.$$  

Then the map $F : S^n \times I \to S^n$ defined by $F(\vec{z}, t) = e^{i\pi t \vec{z}}$ is a homotopy from the identity map to the antipodal map.

If you prefer to work in $\mathbb{R}^{2m}$ you can take

$$F(x_1, y_1, x_2, y_2, \ldots, x_m, y_m, t) = (x_1 \cos \pi t - y_1 \sin \pi t, \ x_1 \sin \pi t + y_1 \cos \pi t, \ldots),$$

but this is less compact.