Solutions to Exam 2

1. (a) If $y = 0$ then $\frac{x}{a} = 1$, so $x = a$ is the x-intercept. If $x = 0$ then $\frac{y}{b} = 1$, so $y = b$ is the y-intercept.

   (b) The square of the distance from a point (x, y) to the origin is $s = x^2 + y^2$. If (x, y) is on the given line, we can solve for y to get $y = b - \frac{b}{a}x$, so

$$s = x^2 + \left(b - \frac{b}{a}x\right)^2.$$

   We wish to choose x to maximize s, so we set $\frac{ds}{dx} = 0$:

$$\begin{aligned} 2x + 2\left(b - \frac{b}{a}x\right)\left(-\frac{b}{a}\right) &= 0 \\ 2x - 2\frac{b^2}{a} + 2\frac{b^2}{a^2}x &= 0 \\ a^2x - ab^2 + b^2x &= 0 \\ (a^2 + b^2)x &= ab^2 \\ x &= \frac{ab^2}{a^2 + b^2} \end{aligned}$$

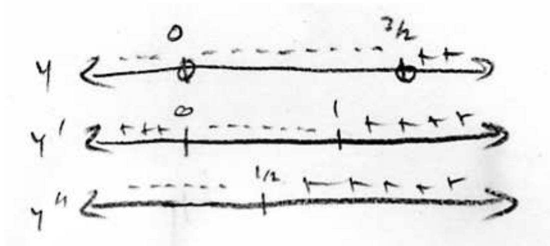
   where in the third line we divided through by 2 and multiplied through by a^2. Thus $y = b - \frac{b}{a}x = \frac{ba^2 + b^3}{a^2 + b^2} - \frac{b^3}{a^2 + b^2} = \frac{ba^2}{a^2 + b^2}$, so the point is

$$\left(\frac{ab^2}{a^2 + b^2}, \frac{a^2b}{a^2 + b^2}\right).$$

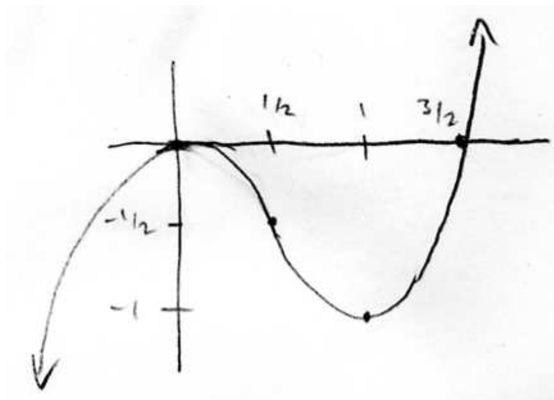
2. This is of the form $\infty - \infty$.

$$\lim_{x \to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x \to 0}\frac{x - \sin x}{x \sin x} = \lim_{x \to 0}\frac{1 - \cos x}{\sin x + x\cos x} = \lim_{x \to 0}\frac{\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0.$$

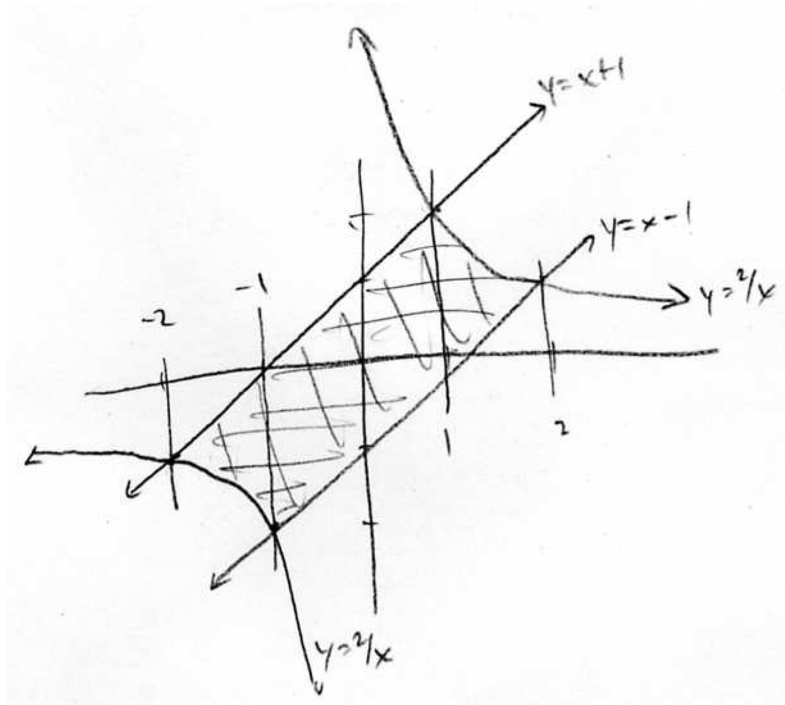
3. $y = 2x^3 - 3x^2 = x^2(2x-3)$, so $y' = 6x^2 - 6x = 6x(x-1)$, so $y'' = 12x - 6 = 6(2x-1)$. When $y = 0$, $x = 0$ or $\frac{3}{2}$. When $y' = 0$, $x = 0$ or 1. When $y'' = 0$, $x = \frac{1}{2}$. The signs of y, y', and y'' are as follows:



When $x = \frac{1}{2}$, $y = \frac{1}{4}(1-3) = -\frac{1}{2}$. When $x = 1$, $y = 1(2-3) = -1$. As $x \to \infty$, $y \to \infty$. As $x \to -\infty$, $y \to -\infty$.



4. (a) The two lines do not intersect. The hyperbola intersects $y = x+1$ when $x+1 = \dfrac{2}{x}$, so $x^2 + x = 2$, so $0 = x^2 + x - 2 = (x+2)(x-1)$, so $x = 1$ or -2. It intersects $y = x - 1$ when $x - 1 = \dfrac{2}{x}$, so $x^2 - x = 2$, so $0 = x^2 - x - 2 = (x-2)(x+1)$, so $x = -1$ or 2. Thus the region looks like this:



The area is

$$\int_{-2}^{-1} \left((x+1) - \frac{2}{x} \right) dx + \int_{-1}^{1} [(x+1) - (x-1)] \, dx + \int_{1}^{2} \left(\frac{2}{x} - (x-1) \right) dx.$$

Alternatively, you could just find the area to the right of the y-axis and double it.

(b)

$$\begin{aligned}
&\int_{-2}^{-1} \left(x + 1 - \frac{2}{x} \right) dx + \int_{-1}^{1} 2 \, dx + \int_{1}^{2} \left(\frac{2}{x} - x + 1 \right) dx \\
&= \left[\frac{x^2}{2} + x - 2\log|x| \right]_{-2}^{-1} + 2x \Big|_{-1}^{1} + \left[2\log|x| - \frac{x^2}{2} + x \right]_{1}^{2} \\
&= \left(\frac{1}{2} - 1 - 0 \right) - (2 - 2 - 2\log 2) + 2(1) - 2(-1) + (2\log 2 - 2 + 2) - \left(0 - \frac{1}{2} + 1 \right) \\
&= 4\log 2 + 3 \approx 5.77
\end{aligned}$$