Exam 3

Sunday, May 7

1. (25 points total) Suppose that $(x^2+5)y' - xy = x$ and when $x = 2$, $y = 0$.
   (a) (10 points) Solve this as an inseparable equation, that is, with an integrating factor. Begin by getting nothing in front of y'. Pay attention to the minus sign.
   (b) (10 points) Solve this as a separable equation. Begin by moving xy to the other side and factoring.
   (c) (5 points) Check that your answer satisfies the original differential equation.

2. (35 points total)
   (a) (5 points) Let $n > 1$ be a constant. Sketch the region in the first quadrant bounded by the curves $y = x^n$ and $y = \sqrt[n]{x} = x^{1/n}$. As a warm-up, you might see what happens when $n = 2$ or $n = 3$.
   (b) (5 points) Argue that you will get the same volume whether you revolve this region around the x- or the y-axis.
   (c) (10 points) Using the disk method, find the volume of the solid generated by revolving this region around the axis of your choice. Your answer will depend on n.
   (d) (10 points) Find the same volume using the shell method. Again, you may choose either axis, independent of your choice in part (c).
   (e) (5 points) What is the limit of your answer as $n \to 1$? As $n \to \infty$? Why does this make sense?

3. (40 points total) According to Torricelli's Law, water drains out of a tank at a rate proportional to the square root of the water's depth. Consider a concial tank of height 1 meter and radius 1 meter.* The tank was initially full and took an hour to drain. At what time was the tank half full? Find the answer as follows:
   (a) (5 points) Draw a picture of the tank when it is somewhere between full and empty. Label something r, something h, and something V.
   (b) (5 points) According to Torricelli's Law, water drains out of a tank at a rate proportional to the square root of the water's depth (i.e. height). Write a differential equation that reflects this fact.
   (c) (1 point) Using similar triangles, find an expression for r in terms of h.
   (d) (2 points) Recall that the volume of a cone is $V = \frac{1}{3}\pi r^2 h$. Find an expression for V in terms of h alone.
   (e) (5 points) Differentiate your answer to part (d) to obtain an expression for dV/dt in terms of h and dh/dt.
   (f) (2 points) From parts (b) and (e) you have two expressions for dV/dt. Set these equal to each other to obtain a differential equation describing the height of the water in the tank.
   (g) (10 points) The tank was initially full and took an hour to drain. Using this initial condition, solve the differential equation you wrote in part (f) to obtain an expression for h in terms of t.
   (h) (5 points) When the tank is half full, what is h?
   (i) (5 points) At what time was the tank half full?

*If you're feeling ambitious, work instead with height H and radius R and conclude that the answer does not depend on the dimensions of the cone.