Solutions to Problem Set 10

I. Problems to be graded on completion.

1. Evaluate the following indefinite integrals:

   a. Let $u = x^3 + 1$, so $du = 3x^2\ dx$, so $x^2\ dx = \frac{1}{3}\ du$.

$$\int x^2\sqrt{x^3+1}\ dx = \int u^{1/2}\tfrac{1}{3}du = \frac{1}{3}\frac{u^{3/2}}{2/3} + C = \tfrac{2}{9}(x^3+1)^{3/2} + C$$

   b. Let $u = e^x$, so $du = e^x\ dx$.

$$\int e^x \cos(e^x)\ dx = \int \cos u\ du = \sin u + C = \sin(e^x) + C$$

   c. Let $u = x^2 + 1$, so $du = 2x\ dx$.

$$\int \frac{2x\ dx}{x^2+1} = \int \frac{du}{u} = \log|u| + C = \log|x^2+1| + C$$

   d. Let $u = \sin x$, so $du = \cos x\ dx$.

$$\int e^{\sin x}\cos x\ dx = \int e^u\ du = e^u + C = e^{\sin x} + C$$

   e. Let $u = x^2 + 1$, so $du = 2x\ dx$, so $3x\ dx = \frac{3}{2}\ du$.

$$\int \frac{3x\ dx}{(x^2+1)^2} = \int \frac{\frac{3}{2}\ du}{u^2} = \int \tfrac{3}{2}u^{-2}\ du = \frac{3}{2}\frac{u^{-1}}{-1} + C = \frac{-3}{2u} + C = \frac{-3}{2x^2+2} + C$$

   f. Let $u = x^{1/4} + 1$, so $du = \frac{1}{4}x^{-3/4}\ dx$, so $x^{-3/4}\ dx = 4\ du$.

$$\int x^{-3/4}(x^{1/4}+1)^{-2}\ dx = \int 4u^{-2}\ du = -4u^{-1} + C = \frac{-4}{x^{1/4}+1} + C$$

   g. Let $u = e^{-2x}$, so $du = -2e^{-2x}\ dx$, so $\dfrac{dx}{e^{2x}} = -\frac{1}{2}\ du$.

$$\int \frac{\tan(e^{-2x})}{e^{2x}}\ dx = \int -\tfrac{1}{2}\tan u\ du = \tfrac{1}{2}\log|\cos u| + C = \tfrac{1}{2}\log|\cos(e^{2x})| + C$$

2. Evaluate the following definite integrals:

   a. Let $u = x^2 + 1$, so $du = 2x\ dx$, so $x\ dx = \frac{1}{2}\ du$. When $x = 0$, $u = 1$. When $x = \sqrt{3}$, $u = 4$.

$$\int_0^{\sqrt{3}} x(x^2+1)^3\ dx = \int_1^4 \tfrac{1}{2}u^3\ du = \left[\frac{1}{2}\frac{u^4}{4}\right]_1^4 = \frac{1}{8}(256-1) = \frac{255}{8}$$

   b. Let $u = x^2 + 16$, so $du = 2x\ dx$, so $x\ dx = \frac{1}{2}\ du$. When $x = 0$, $u = 16$. When $x = 3$, $u = 25$.

$$\int_0^3 \frac{x\ dx}{\sqrt{x^2+16}} = \int_1 6^2 5\tfrac{1}{2}u^{-1/2}\ du = \left[\frac{1}{2}\frac{u^{1/2}}{1/2}\right]_{16}^{25} = \sqrt{25} - \sqrt{16} = 1$$

c. Let $u = \pi x$, so $du = \pi \; dx$, so $dx = \frac{1}{\pi} \; du$. When $x = \frac{1}{4}$, $u = \frac{\pi}{4}$. When $x = \frac{1}{3}$, $x = \frac{\pi}{3}$.

$$\int_{1/4}^{1/3} \sin(\pi x) \; dx = \int_{\pi/4}^{\pi/3} \frac{1}{\pi} \sin u \; du = -\frac{1}{\pi} \cos u \bigg|_{\pi/4}^{\pi/3} = -\frac{1}{\pi} \left(\frac{1}{2} - \frac{\sqrt{2}}{2} \right) = \frac{\sqrt{2} - 1}{2\pi} \approx .066$$

d. Let $u = \cos x$, so $du = -\sin x \; dx$, so $\sin x \; dx = -du$. When $x = 0$, $u = 1$. When $x = \frac{\pi}{2}$, $u = 0$.

$$\int_0^{\pi/2} (\cos x)^2 \sin x \; dx = \int_1^0 -u^2 \; du = \int_0^1 u^2 \; du = \frac{u^3}{3} \bigg|_0^1 = \frac{1}{3}$$

II. Problems to be graded on correctness.

1. Evaluate the following indefinite integrals. Do not take any shortcuts.

a. Let $u = 2x + 3$, so $du = 2 \; dx$, so $du = \frac{1}{2} \; du$.

$$\int (2x+3)^2 \; dx = \int \frac{1}{2} u^2 \; du = \frac{1}{2} \frac{u^3}{3} + C = \frac{1}{2} \frac{(2x+3)^3}{3} + C$$

b. Let $u = -4x + 2$, so $du = -4 \; dx$, so $du = -\frac{1}{4} \; du$.

$$\int e^{-4x+2} \; dx = \int -\frac{1}{4} e^u \; du = -\frac{1}{4} e^u + C = -\frac{1}{4} e^{-4x+2} + C$$

c. Let $u = x - 7$, so $du = dx$.

$$\int \sin(x-7) \; dx = \int \sin u \; du = -\cos u + C = -\cos(x-7) + C$$

d. Let $u = \frac{1}{6}x - 1$, so $du = \frac{1}{6} \; dx$, so $dx = 6 \; du$.

$$\int \left(\sec(\tfrac{1}{6}x - 1)\right)^2 \; dx = \int 6(\sec u)^2 \; du = 6 \tan u + C = 6 \tan(\tfrac{1}{6}x - 1) + C$$

e. Let $u = 17x$, so $du = 17 \; dx$, so $dx = \frac{1}{17} \; du$.

$$\int \tan(17x) \; dx = \int \frac{1}{17} \tan u \; du = -\frac{1}{17} \log|\cos u| + C = -\frac{1}{17} \log|\cos(17x)| + C$$

f. Let $u = \frac{3}{2}x + 5$, so $du = \frac{3}{2} \; dx$, so $dx = \frac{2}{3} \; du$.

$$\int \left(\tfrac{3}{2}x + 5\right)^{1/2} \; dx = \int \frac{2}{3} u^{1/2} \; du = \frac{2}{3} \frac{u^{3/2}}{3/2} + C = \frac{2}{3} \frac{\left(\frac{3}{2}x + 5\right)^{3/2}}{3/2} + C$$

In general, if $\int f(x) \; dx = F(x) + C$ then $\int f(ax+b) \; dx = \frac{1}{a} F(ax+b) + C$. We might like to say that we take the anti-derivative of the outside divided by the derivative of the inside, but this does not work if our substitution is not linear. For example,

$$\int \cos(x^2) \; dx \neq \frac{1}{2x} \sin(x^2) + C$$

because the derivative of $\frac{1}{2}x^{-1} \sin(x^2)$ is $\cos(x^2) - \frac{1}{2}x^{-2} \sin(x^2)$, which is different from $\cos(x^2)$. In particular, there is nothing as nice as the chain rule for anti-derivatives.

2.

$$\int \frac{x^3 - 1}{x^2} \; dx = \int (x - x^{-2}) \; dx = \frac{x^2}{2} - \frac{x^{-1}}{-1} + C = \frac{x^2}{2} + \frac{1}{x} + C = \frac{x^3 + 2}{2x} + C$$