Solutions to Problem Set 11

I. Problems to be graded on completion.

1. a. Consider the line $2x + y = 4$. If $x = 0$ then $y = 4$, so the y-intercept is 4. If $y = 0$ then $2x = 4$, so $x = 2$, so the x-intercept is 2.

![Line Graph](image)

Around the $x$-axis, the radius of a slice is $4 - 2x$, so

$$
\int_0^2 \pi (4x - 2)^2 \, dx = \pi \int_0^2 (16x^2 - 16x + 4) \, dx = \pi \left[ \frac{16}{3} x^3 - 8x^2 + 4x \right]_0^2 = \frac{56\pi}{3} \approx 58.6.
$$

Around the $y$-axis, the radius of a slice is $2 - \frac{1}{2}y$, so

$$
\int_0^4 \pi \left( 2 - \frac{1}{2}x \right)^2 \, dx = \pi \int_0^4 \left( 4 - 2x + \frac{1}{4}x^2 \right) \, dx = \pi \left[ 4x - x^2 + \frac{1}{12}x^3 \right]_0^4 = \frac{16\pi}{3} \approx 16.8.
$$

b. These intersect when $x^2 = x^{1/3}$, so $x^6 = x$, so $x^6 - x = 0$. so $x(x^5 - 1) = 0$, so $x = 0$ or $x^5 = 1$, so $x = 1$. In between, which one is on top? If $x = \frac{1}{4}$, $x^2 = \frac{1}{64}$ while $x^{1/3} = \frac{1}{2}$, so $x^{1/3}$ is above $x^2$ on the interval $(0, 1)$.

![Intersection Graph](image)

Around the $x$-axis, the outer radius is $x^{1/3}$ and the inner radius is $x^2$, so

$$
\int_0^1 \pi x^{2/3} \, dx - \int_0^1 \pi x^4 \, dx = \frac{3}{5} \pi x^{5/3} \bigg|_0^1 - \frac{1}{5} \pi x^5 \bigg|_0^1 = \frac{2\pi}{5} \approx 1.26.
$$

Around the $y$-axis, let’s use the shell method. The radius of the shell is $x$ and the height is $x^{1/3} - x^2$, so

$$
\int_0^1 2\pi x(x^{1/3} - x^2) \, dx = 2\pi \int_0^1 (x^{4/3} - x^3) \, dx = 2\pi \left[ \frac{3}{7} x^{7/3} - \frac{1}{4} x^4 \right]_0^1 = \frac{5\pi}{14} \approx 1.12.
$$
c. The region looks like this:

![Diagram of a region](image)

Around the x-axis, the radius of a slice is \( \sin x \), so

\[
\int_{\pi/4}^{\pi/2} \pi(\sin x)^2 \, dx = \frac{\pi}{2} \int_{\pi/4}^{\pi/2} (1 - \cos 2x) \, dx = \frac{\pi}{2} \left[ x - \frac{1}{2} \sin 2x \right]_{\pi/4}^{\pi/2} = \frac{\pi}{2} \left( \frac{\pi}{4} + \frac{1}{2} \right) \approx 2.02.
\]

Around the y-axis, let’s use the shell method again. The radius is \( x \) and the height is \( \sin x \), so

\[
\int_{\pi/4}^{\pi/2} 2\pi x \sin x \, dx
\]

would give us the number we wanted if we could do the integral.

2. a. 

\[
8 \int_0^2 (\sqrt{4 - x^2})^2 \, dx = 8 \int_0^2 (4 - x^2) \, dx = 8 \left[ 4x - \frac{1}{3} x^3 \right]_0^2 = \frac{128}{3} = 42 + \frac{2}{3}
\]

b. Consider half of an equilateral triangle. If the base is \( a \), the hypotenuse is \( 2a \), so the height is \( \sqrt{3}a \), so the area is \( \frac{\sqrt{3}}{2} a^2 \).

\[
4 \int_0^2 \frac{\sqrt{3}}{2} (\sqrt{4 - x^2})^2 \, dx = 2\sqrt{3} \int_0^2 (4 - x^2) \, dx = \frac{32}{\sqrt{3}} \approx 18.5
\]

c. If the base of the triangle is \( 2\sqrt{4 - x^2} \) and the height is 3, the area is \( 3\sqrt{4 - x^2} \).

\[
2 \int_0^2 3\sqrt{4 - x^2} \, dx = 6 \int_0^2 \sqrt{4 - x^2} \, dx = 6\pi
\]

where we interpreted \( \int_0^2 \sqrt{4 - x^2} \, dx \) as the area of a quarter-circle as in problem set 9.

8.

\[
y = \frac{2}{3}(x^2 + 1)^{3/2}
\]

\[
y' = (x^2 + 1)^{1/2}(2x)
\]

\[
(y')^2 = (x^2 + 1)(2x)^2 = 4x^4 + 4x^2
\]

\[
1 + (y')^2 = 4x^4 + 4x^2 + 1 = (2x^2 + 1)^2
\]

so

\[
\int_1^2 \sqrt{1 + (y')^2} \, dx = \int_1^2 (2x^2 + 1) \, dx = \left[ \frac{2}{3}x^3 + x \right]_1^2 = \frac{17}{3} = 5 + \frac{2}{3}
\]

9.

\[
y = (4 - x^{2/3})^{3/2}
\]

\[
y' = \frac{3}{2}(4 - x^{2/3})^{1/2}(-\frac{2}{3}x^{-1/3}) = -x^{1/3}(4 - x^{2/3})^{1/2}
\]

\[
(y')^2 = x^{-2/3}(4 - x^{2/3}) = 4x^{-2/3} - 1
\]

\[
1 + (y')^2 = 4x^{-2/3} = (2x^{-1/3})^2
\]
so
\[ \int_1^8 \sqrt{1 + (y')^2} \, dx = \int_1^8 2x^{-1/3} \, dx = 3x^{2/3} \bigg|_1^8 = 9 \]

II. Problems to be graded on correctness.

1. The radius of a slice is \( \sqrt{r^2 - x^2} \), so
\[ \int_h^r \pi(r^2 - x^2) \, dx = \pi \left[ r^2x - \frac{1}{3}x^3 \right]_h^r = \pi \left( \frac{2}{3}r^3 - r^2h + \frac{1}{3}h^3 \right) . \]

27. The base is a quarter-circle \( y = \sqrt{1 - x^2} \) or \( x = \sqrt{1 - y^2} \). The cross-sections are squares.
\[ \int_0^1 (\sqrt{1 - y^2})^2 \, dy = \int_0^1 (1 - y^2) \, dy = \left[ y - \frac{1}{3}y^3 \right]_0^1 = \frac{2}{3} \]

2. \( y = \frac{1}{2}e^x + \frac{1}{2}e^{-x} \)
\( y' = \frac{1}{2}e^x - \frac{1}{2}e^{-x} \)
\( (y')^2 = \frac{1}{4}e^{2x} - \frac{1}{2} + \frac{1}{4}e^{-2x} \)
\[ 1 + (y')^2 = \frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x} = \left( \frac{1}{2}e^x + \frac{1}{2}e^{-x} \right)^2 \]
so
\[ \int_{-1}^1 \sqrt{1 + (y')^2} \, dx = \int_{-1}^1 \left( \frac{1}{2}e^x + \frac{1}{2}e^{-x} \right) \, dx = \left[ \frac{1}{2}e^x - \frac{1}{2}e^{-x} \right]_{-1}^1 = e - \frac{1}{e} \approx 2.35 \]

3. a. \(-3 + 1 = -2\), not \(-4\).
\[ \frac{d}{dx} (-\frac{1}{4}x^{-4} + C) = x^{-5} \]

b. The integral of a product is not the product of the integrals.
\[ \frac{d}{dx} \left( \frac{1}{3}x^3 \tan x + C \right) = x^2 \tan x + \frac{1}{3}x^3(\sec x)^2 \]

c. There is no chain rule for integrals.
\[ \frac{d}{dx} \left[ \sin(x^3 + 1) \left( \frac{x^4}{4} + x \right) + C \right] = \cos(x^3 + 1)(3x^2) \left( \frac{x^4}{4} + x \right) + \sin(x^3 + 1)(x^3 + 1) \]

d. The integral of \( \frac{1}{\text{whatever}} \) is not automatically \( \log |\text{whatever}| + C \). This only works if whatever = \( x + a \) for some number \( a \).
\[ \frac{d}{dx} (\log |4x^2 + 1| + C) = \frac{8x}{4x^2 + 1} \]