Solutions to Problem Set 1

I. Problems to be graded on completion.

Graphs

1. (a) 2. (b) -1. (c) does not exist. (d) -3.
2. (a) -4. (b) -4. (c) -4. (d) 2.
4. (a) 1. (b) ∞. (c) does not exist. (d) 1.
5. (a) $-\infty$. (b) $-\infty$. (c) $-\infty$. (d) 1.
8. (a) 2. (b) 2. (c) 2. (d) 2.

§2.4

8. $\lim_{t\to -7} \dfrac{t^2+4t-21}{t+7} = \lim_{t\to -7} \dfrac{(t+7)(t-3)}{t+7} = \lim_{t\to -7}(t-3) = -10.$

10. $\lim_{x\to 0} \dfrac{x^4+2x^3-x^2}{x^2} = \lim_{x\to 0}(x^2+2x-1) = -1.$

14. $\lim_{x\to 7^+} \dfrac{\sqrt{(t-7)^3}}{t-7} = \lim_{x\to 7^+} \dfrac{(t-7)^{3/2}}{(t-7)^1} = \lim_{x\to 7^+}(t-7)^{1/2} = 0.$

§2.6

6.

$$\lim_{x\to -3} \frac{4x^3+1}{7-2x^2} \overset{7}{=} \frac{\lim_{x\to -3}(4x^3+1)}{\lim_{x\to -3}(7-2x^2)} \overset{4}{\underset{5}{=}} \frac{\lim_{x\to -3} 4x^3 + \lim_{x\to -3} 1}{\lim_{x\to -3} 7 - \lim_{x\to -3} 2x^2}$$

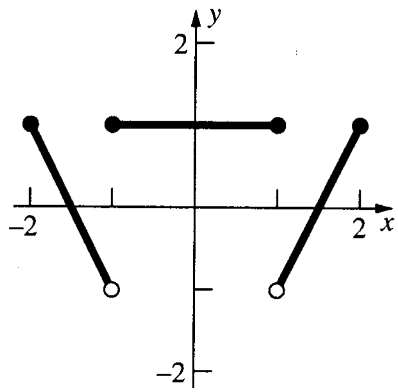
$$\overset{1,3}{=} \frac{4\lim_{x\to -3} x^3 + 1}{7 - 2\lim_{x\to -3} x^2} \overset{8}{=} \frac{4\left(\lim_{x\to -3} x\right)^3 + 1}{7-2\left(\lim_{x\to -3} x\right)^2} \overset{2}{=} \frac{4(-3)^3+1}{7-2(-3)^2} = \frac{107}{11}$$

22. $\lim_{x\to a} \dfrac{2f(x)-3g(x)}{f(x)+g(x)} = \dfrac{2\cdot 3 - 3\cdot(-1)}{3+(-1)} = \dfrac{9}{2}.$

§2.9

28. The function is continuous on $(5,\infty)$ and not defined on $(-\infty,5]$.
30. The function is continuous everywhere.
32. The function is continuous everywhere.

36. There are many possible solutions. One is



II. Problems to be graded on correctness.

1. First we simplify:

$$\begin{aligned}
\frac{\sqrt{x}-2}{(x-4)^2} - \frac{1}{x^2-4x} &= \frac{\sqrt{x}-2}{(\sqrt{x}+2)(\sqrt{x}-2)(x-4)} - \frac{1}{x(x-4)} \\
&= \frac{1}{(\sqrt{x}+2)(x-4)} - \frac{1}{x(x-4)} \\
&= \frac{x-(\sqrt{x}+2)}{x(\sqrt{x}+2)(x-4)} \\
&= \frac{(\sqrt{x})^2-\sqrt{x}-2}{x(\sqrt{x}+2)(x-4)} \\
&= \frac{(\sqrt{x}-2)(\sqrt{x}+1)}{x(\sqrt{x}+2)(\sqrt{x}+2)(\sqrt{x}-2)} \\
&= \frac{\sqrt{x}+1}{x(\sqrt{x}+2)^2}.
\end{aligned}$$

Now
$$\lim_{x\to 4}\left[\frac{\sqrt{x}-2}{(x-4)^2} - \frac{1}{x^2-4x}\right] = \lim_{x\to 4}\frac{\sqrt{x}+1}{x(\sqrt{x}+2)^2} = \frac{\sqrt{4}+1}{4(\sqrt{4}+2)^2} = \frac{3}{64}.$$

2. The function, call it $f(x)$, is continuous everywhere except at -3, 0, 2, and 6. In particular, it is continuous at 5. At -3, $\lim_{x\to -3} f(x) = 2$, but $f(-3)$ is undefined. At 0, $f(0) = 3$ and f is continuous from the positive side, but $\lim_{x\to 0^-} f(x) = 0$. At 2, $\lim_{x\to 2} f(x) = 1$, but $f(2) = -1$. At 6, $\lim_{x\to 6} f(x) = -\infty$, but $f(6)$ is undefined.
3. There is an $\epsilon > 0$ such that for every $\delta > 0$ there is an x such that $0 < |x-c| < \delta$ but $|f(x)-L| \geq \epsilon$.
4. Suppose on the contrary that $x \neq y$. Then $|x-y| \neq 0$, and absolute values cannot be negative, so $|x-y| > 0$. Then $|x-y| > |x-y|/2$, since any positive number is greater than half of itself. But now if we let $\epsilon = |x-y|/2$, we have an $\epsilon > 0$ for which $|x-y| > \epsilon$. But $|x-y|$ is less than *every* positive number ϵ, so our assumption that $x \neq y$ must have been false. Thus $x = y$.

5.

(a)

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}.
$$

(b)

$$
\begin{aligned}
(x+y)^2 &= (x+y)(x+y) \\
&= x(x+y) + y(x+y) \\
&= (x^2 + xy) + (xy + y^2) \\
&= \mathbf{x^2 + 2xy + y^2}.
\end{aligned}
$$

$$
\begin{aligned}
(x+y)^3 &= (x+y)(x+y)^2 \\
&= (x+y)(x^2 + 2xy + y^2) \\
&= x(x^2 + 2xy + y^2) + y(x^2 + 2xy + y^2) \\
&= (x^3 + 2x^2y + xy^2) + (x^2y + 2xy^2 + y^3) \\
&= \mathbf{x^3 + 3x^2y + 3xy^2 + y^3}.
\end{aligned}
$$

$$
\begin{aligned}
(x+y)^4 &= (x+y)(x+y)^3 \\
&= (x+y)(x^3 + 3x^2y + 3xy^2 + y^3) \\
&= x(x^3 + 3x^2y + 3xy^2 + y^3) + y(x^3 + 3x^2y + 3xy^2 + y^3) \\
&= (x^4 + 3x^3y + 3x^2y^2 + xy^3) + (x^3y + 3x^2y^2 + 3xy^3 + y^4) \\
&= \mathbf{x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4}.
\end{aligned}
$$

(c) $(x+y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 6xy^6 + y^7$.