I. Problems to be graded on completion.

1. Substitute $u = 4x$ and $v = 2x$. As $x \to 0$, $u \to 0$ and $v \to 0$.

\[
\lim_{x \to 0} \frac{4x}{2x} = \lim_{x \to 0} \frac{4x}{2x} = \left( \lim_{x \to 0} \frac{4x}{2x} \right) \left( \lim_{u \to 0} \frac{\sin u}{u} \right) = 2 \cdot 1 = 2
\]

2. Substitute $u = x^2$. As $x \to 0$, $u \to 0$.

\[
\lim_{x \to 0} \frac{\sin x^2}{x^2} = \lim_{x \to 0} \frac{x \sin x^2}{x^2} = \left( \lim_{x \to 0} x \right) \left( \lim_{x \to 0} \frac{\sin x^2}{x^2} \right) = 0 \cdot \lim_{u \to 0} \frac{\sin u}{u} = 0 \cdot 1 = 0
\]

3. Substitute $u = 3x$. As $x \to 0$, $u \to 0$.

\[
\lim_{x \to 0} \frac{(\sin 3x)^2}{5x^2} = \lim_{x \to 0} \frac{1}{5} \left( \frac{\sin 3x}{x} \right)^2 = \frac{1}{5} \left( \lim_{x \to 0} \frac{\sin 3x}{x} \right)^2 = \frac{1}{5} \left( 3 \lim_{x \to 0} \frac{\sin 3x}{3x} \right)^2
\]

\[
= \frac{1}{5} \left( 3 \lim_{u \to 0} \frac{\sin u}{u} \right)^2 = \frac{1}{5} (3 \cdot 1)^2 = \frac{9}{5}
\]

4. 

\[
\lim_{x \to 0} \frac{\sin x}{\frac{\pi}{3}} = \frac{\sin \frac{\pi}{3}}{\frac{\pi}{3}} = \frac{\sqrt{3}}{3} = \frac{3\sqrt{3}}{2\pi}
\]

12. 

\[
\lim_{x \to \infty} \sqrt[3]{\frac{\pi x^3 + 3x}{\sqrt{2x^3} + 7x}} = \sqrt[3]{\lim_{x \to \infty} \frac{\pi x^3 + 3x}{\sqrt{2x^3} + 7x}} = \sqrt[3]{\lim_{x \to \infty} \frac{\pi + \frac{3}{x}}{\sqrt{2} + \frac{7}{x}}} = \sqrt[3]{\frac{\pi + 0}{\sqrt{2} + 0}} = \frac{\sqrt{\pi}}{\sqrt{2}}
\]

14. 

\[
\lim_{x \to \infty} \sqrt{\frac{x^2 + x + 3}{(x-1)(x+1)}} = \sqrt{\lim_{x \to \infty} \frac{x^2 + x + 3}{x^2 - 1}} = \sqrt{\lim_{x \to \infty} \frac{1 + \frac{1}{x} + \frac{3}{x^2}}{1 - \frac{1}{x}}} = \sqrt{\frac{1 + 0 + 0}{1 - 0}} = 1
\]

16. 

\[
\lim_{x \to \infty} \frac{\sqrt{2x + 1}}{x + 4} = \lim_{x \to \infty} \frac{\frac{1}{2}\sqrt{2x + 1} + \frac{1}{2}}{1 + \frac{4}{x}} = \lim_{x \to \infty} \frac{\frac{1}{2}(2x + 1)}{1 + \frac{4}{x}} = \lim_{x \to \infty} \frac{\sqrt{\frac{2 + \frac{1}{2x}}{1 + \frac{4}{x}}}}{1 + 0} = \frac{\sqrt{0 + 0}}{1 + 0} = 0
\]

18. 

\[
\lim_{x \to \infty} \sqrt{x^2 + 2x - x} = \lim_{x \to \infty} \frac{(\sqrt{x^2 + 2x - x})(\sqrt{x^2 + 2x + x})}{\sqrt{x^2 + 2x + x}} = \lim_{x \to \infty} \frac{(x^2 + 2x - x)^2}{\sqrt{x^2 + 2x + x}} = \lim_{x \to \infty} \frac{2x}{\sqrt{x^2 + 2x + x}} = \lim_{x \to \infty} \frac{2}{\sqrt{\frac{1 + \frac{4}{x}}{1 + 0}} = \frac{2}{\sqrt{1 + 0 + 0}} = 2}
\]
II. Problems to be graded on correctness.

21. If we plug in $x = 4$, we get $\frac{4}{5}$, so the limit is either $\infty$ or $-\infty$. If $x$ is a little bit bigger than $4$ then $x - 4$ is a little bit bigger than $0$, so $\frac{x}{x-4}$ is positive. Thus

$$\lim_{x \to 4^+} \frac{x}{x-4} = \infty.$$  

23. If we plug in $t = 3$, we get $\frac{9}{7}$, so the limit is either $\infty$ or $-\infty$. If $t$ is a little bit less than $3$ then $t^2$ is a little bit less than $9$ and $9 - t^2$ is a little bit bigger than $0$, so $\frac{t^2}{9-t^2}$ is positive. Thus

$$\lim_{t \to 3^-} \frac{t^2}{9-t^2} = \infty.$$  

25. If we plug in $x = 5$, we get $\frac{25}{6}$, so the limit is either $\infty$ or $-\infty$. If $x$ is a little bit less than $5$ then $x^2$ is a little bit less than $25$, $x - 5$ is a little bit less than $0$, and $3 - x$ is a little bit bigger than $-2$, so $\frac{x^2}{(x-5)(3-x)}$ is positive. Thus

$$\lim_{x \to 5^-} \frac{x^2}{(x-5)(3-x)} = \infty.$$  

29.  

$$\lim_{x \to 3^-} \frac{x^2 - x - 6}{x - 3} = \lim_{x \to 3^-} \frac{(x - 3)(x + 2)}{x - 3} = \lim_{x \to 3^-} (x + 2) = 5$$  

50. If $v$ is a little bit less than $c$ then $v^2/c^2$ is a little bit less than $1$, so $1 - v^2/c^2$ is a little bit bigger than $0$, and $m_0$ has to be positive, so $\frac{m_0}{\sqrt{1-v^2/c^2}}$ is positive. Thus

$$\lim_{v \to c^-} \frac{m_0}{\sqrt{1-v^2/c^2}} = \infty.$$  

4.  

$$e^{-2\log x} = (e^{\log x})^{-2} = x^{-2} = \frac{1}{x^2}.$$  

6.  

$$\log e^{-2x-3} = -2x - 3.$$  

8.  

$$e^{x-\log x} = \frac{e^x}{e^{\log x}} = \frac{e^x}{x}.$$  

10.  

$$e^{\log x^2-y \log x} = e^{2\log x-y \log x} = e^{(\log x)(2-y)} = (e^{\log x})^{2-y} = x^{2-y}.$$  

II. Problems to be graded on correctness.

1. Substitute $u = 3x$. As $x \to 0$, $u \to 0$.

$$\lim_{x \to 0} \frac{\tan 3x}{2x^2 + 5x} = \lim_{x \to 0} \frac{\sin 3x}{\cos 3x} \cdot \frac{\cos 3x}{(2x + 5)x} = \lim_{x \to 0} \frac{1}{(2x + 5)} \cdot \frac{\sin 3x}{x} = \left(\lim_{x \to 0} \frac{3}{(2x + 5)} \right) \cdot \frac{\sin 3x}{3x} \cdot \left(\lim_{u \to 0} \frac{\sin u}{u}\right) = \frac{3}{5} \cdot \frac{3}{5} = \frac{3}{5}$$  

2. Observe that $-1 \leq \sin x \leq 1$ for all $x$. For $x$ large (so we don’t have to worry about negative numbers flipping our inequality), we have

$$-\frac{1}{x} \leq \frac{-\sin x}{x} \leq \frac{1}{x}.$$  

Now $\lim_{x \to \infty} -\frac{1}{x} = \lim_{x \to \infty} \frac{1}{x} = 0$, so by the sandwich theorem, $\lim_{x \to \infty} \frac{\sin x}{x} = 0$.  

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3. Substitute \( n = \frac{1}{h} \). As \( h \to 0^+ \), \( n \to \infty \).

\[
\lim_{h \to 0^+} (1 + hx)^{1/h} = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x
\]

4. a. 

\[
2 \log x - 4 \log \frac{1}{y} - 3 \log(xy) = \log \left(\frac{x^2}{y^4 (xy)^3}\right) = \log \frac{y}{x}
\]

b. 

\[
2 \log x - 4 \log \frac{1}{y} - 3 \log(xy) = 2 \log x + 4 \log y - 3 \log x - 3 \log y = \log y - \log x = \log \frac{y}{x}
\]

5. a. There are many possible answers. One is

\[
p(x) = 2x^3 - 17x + 1
\]

\[
q(x) = 5x^3 + x^2 - 8
\]

\[
r(x) = -x^4 + 17
\]

b. 

\[
\lim_{x \to \infty} \frac{2x^3 - 17x + 1}{-x^3 + 17} = \lim_{x \to \infty} \frac{2 - \frac{17}{x^2} + \frac{1}{x^3}}{-1 + \frac{17}{x^2}} = 0 - 0 + 0 = 0
\]

c. 

\[
\lim_{x \to \infty} \frac{2x^3 - 17x + 1}{5x^3 + x^2 - 8} = \lim_{x \to \infty} \frac{2 - \frac{17}{x^2} + \frac{1}{x^3}}{5 + \frac{1}{x} - \frac{8}{x^2}} = \frac{2 - 0 + 0}{5 + 0 - 0} = \frac{2}{5}
\]

d. 

\[
\lim_{x \to \infty} \frac{-x^4 + 17}{5x^3 + x^2 - 8} = \lim_{x \to \infty} \frac{-x + \frac{17}{x^2}}{5 + \frac{1}{x} - \frac{8}{x^3}} = \frac{-\infty - 0 + 0}{5 + 0 - 0} = -\infty
\]

e. We wish to consider an arbitrary rational function \( f(x) \). Let \( m \) and \( n \) be the degrees of the numerator and the denominator and \( a \) and \( b \) be the leading coefficients—that is,

\[
f(x) = \frac{ax^m + \text{lower order terms}}{bx^n + \text{lower order terms}}
\]

In general,

\[
\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} \frac{ax^m}{bx^n}.
\]

To prove this, we divide the top and bottom by \( x^n \) and consider several cases.

- If \( m < n \), \( \lim_{x \to \pm\infty} f(x) = 0 \).
- If \( m = n \), \( \lim_{x \to \pm\infty} f(x) = \frac{a}{b} \).
- If \( m > n \), the limit is \( \pm\infty \), but we have to consider several subcases to say which:
  - If \( \frac{a}{b} \) is positive, \( \lim_{x \to \infty} f(x) = \infty \). If the numerator contains terms of degree greater than \( n \) but less than \( m \), we should check that they do not change our answer. For example,
    \[
    \lim_{x \to \infty} \frac{x^5 - 1000x^4 + 1}{x^3 + 1} = \lim_{x \to \infty} \frac{x^2 - 1000x + \frac{1}{x^3}}{1 + \frac{1}{x^7}}.
    \]
    On the one hand, \( x^2 \to \infty \), but \( -1000x \to -\infty \), so when we put them together, what happens? In fact, the highest-degree term \( (x^2) \) in this case will always win out, but showing this carefully would require some more work.
  - If \( \frac{a}{b} \) is negative, \( \lim_{x \to \infty} f(x) = -\infty \).
  - If \( m > n \) and \( m - n \) is even, \( \lim_{x \to -\infty} f(x) = \infty \) or \( -\infty \) according as \( \frac{a}{b} \) is positive or negative. If \( m - n \) is odd, the reverse is true.

In all cases, our general claim (*) is true.