Solutions to Problem Set 2

I. Problems to be graded on completion.

1. Substitute $u = 4x$ and $v = 2x$. As $x \to 0$, $u \to 0$ and $v \to 0$.

$$\lim_{x\to 0} \frac{\sin 4x}{\sin 2x} = \lim_{x\to 0} \frac{4x \frac{\sin 4x}{4x}}{2x \frac{\sin 2x}{2x}} = \left(\lim_{x\to 0} \frac{4x}{2x}\right) \frac{\left(\lim_{x\to 0} \frac{\sin 4x}{4x}\right)}{\left(\lim_{x\to 0} \frac{\sin 2x}{2x}\right)} = \left(\lim_{x\to 0} \frac{4}{2}\right) \frac{\left(\lim_{u\to 0} \frac{\sin u}{u}\right)}{\left(\lim_{v\to 0} \frac{\sin v}{v}\right)} = \frac{4}{2} \cdot \frac{1}{1} = 2$$

2. Substitute $u = x^2$. As $x \to 0$, $u \to 0$.

$$\lim_{x\to 0} \frac{\sin x^2}{x} = \lim_{x\to 0} \frac{x \sin x^2}{x^2} = \left(\lim_{x\to 0} x\right) \left(\lim_{x\to 0} \frac{\sin x^2}{x^2}\right) = 0 \cdot \lim_{u\to 0} \frac{\sin u}{u} = 0 \cdot 1 = 0$$

3. Substitute $u = 3x$. As $x \to 0$, $u \to 0$.

$$\lim_{x\to 0} \frac{(\sin 3x)^2}{5x^2} = \lim_{x\to 0} \frac{1}{5} \left(\frac{\sin 3x}{x}\right)^2 = \frac{1}{5} \left(\lim_{x\to 0} \frac{\sin 3x}{x}\right)^2 = \frac{1}{5} \left(3 \lim_{x\to 0} \frac{\sin 3x}{3x}\right)^2$$
$$= \frac{1}{5} \left(3 \lim_{u\to 0} \frac{\sin u}{u}\right)^2 = \frac{1}{5}(3 \cdot 1)^2 = \frac{9}{5}$$

4.

$$\lim_{x\to \frac{\pi}{3}} \frac{\sin x}{x} = \frac{\sin \frac{\pi}{3}}{\frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{\frac{\pi}{3}} = \frac{3\sqrt{3}}{2\pi}$$

12.

$$\lim_{x\to\infty} \sqrt[3]{\frac{\pi x^3 + 3x}{\sqrt{2}x^3 + 7x}} = \sqrt[3]{\lim_{x\to\infty} \frac{\pi x^3 + 3x}{\sqrt{2}x^3 + 7x}} = \sqrt[3]{\lim_{x\to\infty} \frac{\pi + \frac{3}{x^2}}{\sqrt{2} + \frac{7}{x^2}}} = \sqrt[3]{\frac{\pi + 0}{\sqrt{2} + 0}} = \frac{\sqrt[3]{\pi}}{\sqrt[6]{2}}$$

14.

$$\lim_{x\to\infty} \sqrt{\frac{x^2 + x + 3}{(x-1)(x+1)}} = \sqrt{\lim_{x\to\infty} \frac{x^2 + x + 3}{x^2 - 1}} = \sqrt{\lim_{x\to\infty} \frac{1 + \frac{1}{x} + \frac{3}{x^2}}{1 - \frac{1}{x^2}}} = \sqrt{\frac{1 + 0 + 0}{1 - 0}} = 1$$

16.

$$\lim_{x\to\infty} \frac{\sqrt{2x+1}}{x+4} = \lim_{x\to\infty} \frac{\frac{1}{x}\sqrt{2x+1}}{1 + \frac{4}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^2}(2x+1)}}{1 + \frac{4}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{2}{x} + \frac{1}{x^2}}}{1 + \frac{4}{x}} = \frac{\sqrt{0+0}}{1+0} = 0$$

18.

$$\lim_{x\to\infty} \sqrt{x^2 + 2x} - x = \lim_{x\to\infty} \frac{(\sqrt{x^2+2x} - x)(\sqrt{x^2+2x} + x)}{\sqrt{x^2+2x} + x} = \lim_{x\to\infty} \frac{(x^2 + 2x) - x^2}{\sqrt{x^2+2x} + x}$$
$$= \lim_{x\to\infty} \frac{2x}{\sqrt{x^2+2x} + x} = \lim_{x\to\infty} \frac{2}{\frac{1}{x}\sqrt{x^2+2x} + 1} = \lim_{x\to\infty} \frac{2}{\sqrt{\frac{1}{x^2}(x^2+2x)} + 1}$$
$$= \lim_{x\to\infty} \frac{2}{\sqrt{1 + \frac{2}{x}} + 1} = \frac{2}{\sqrt{1+0} + 1} = 2$$

21. If we plug in $x = 4$, we get $\frac{4}{0}$, so the limit is either ∞ or $-\infty$. If x is a little bit bigger than 4 then $x - 4$ is a little bit bigger than 0, so $\frac{x}{x-4}$ is positive. Thus

$$\lim_{x \to 4^+} \frac{x}{x-4} = \infty.$$

23. If we plug in $t = 3$, we get $\frac{9}{0}$, so the limit is either ∞ or $-\infty$. If t is a little bit less than 3 then t^2 is a little bit less than 9 and $9 - t^2$ is a little bit bigger than 0, so $\frac{t^2}{9-t^2}$ is positive. Thus

$$\lim_{t \to 3^-} \frac{t^2}{9-t^2} = \infty.$$

25. If we plug in $x = 5$, we get $\frac{25}{0}$, so the limit is either ∞ or $-\infty$. If x is a little bit less than 5 then x^2 is a little bit less than 25, $x - 5$ is a little bit less than 0, and $3 - x$ is a little bit bigger than -2, so $\frac{x^2}{(x-5)(3-x)}$ is positive. Thus

$$\lim_{x \to 5^-} \frac{x^2}{(x-5)(3-x)} = \infty.$$

29.

$$\lim_{x \to 3^-} \frac{x^2 - x - 6}{x - 3} = \lim_{x \to 3^-} \frac{(x-3)(x+2)}{x-3} = \lim_{x \to 3^-} (x+2) = 5$$

50. If v is a little bit less than c then v^2/c^2 is a little bit less than 1, so $1 - v^2/c^2$ is a little bit bigger than 0, and m_0 has to be positive, so $\frac{m_0}{\sqrt{1-v^2/c^2}}$ is positive. Thus

$$\lim_{v \to c^-} \frac{m_0}{\sqrt{1-v^2/c^2}} = \infty.$$

4. $e^{-2\log x} = (e^{\log x})^{-2} = x^{-2} = \dfrac{1}{x^2}$.
6. $\log e^{-2x-3} = -2x - 3$.
8. $e^{x - \log x} = \dfrac{e^x}{e^{\log x}} = \dfrac{e^x}{x}$.
10. $e^{\log x^2 - y \log x} = e^{2\log x - y \log x} = e^{(\log x)(2-y)} = (e^{\log x})^{2-y} = x^{2-y}$.

- $$\lim_{x \to \infty} \Big[\log(x+1) - \log x\Big] = \lim_{x \to \infty} \left[\log \frac{x+1}{x}\right] = \log\left[\lim_{x \to \infty} \frac{x+1}{x}\right] = \log\left[\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)\right] = \log(1+0) = 0$$

II. Problems to be graded on correctness.

1. Substitute $u = 3x$. As $x \to 0$, $u \to 0$.

$$\lim_{x \to 0} \frac{\tan 3x}{2x^2 + 5x} = \lim_{x \to 0} \frac{\frac{\sin 3x}{\cos 3x}}{(2x+5)x} = \lim_{x \to 0} \left(\frac{1}{(2x+5)\cos 3x} \frac{\sin 3x}{x}\right)$$
$$= \left(\lim_{x \to 0} \frac{3}{(2x+5)\cos 3x}\right)\left(\lim_{x \to 0} \frac{\sin 3x}{3x}\right) = \frac{3}{(2 \cdot 0 + 5) \cdot 1}\left(\lim_{u \to 0} \frac{\sin u}{u}\right) = \frac{3}{5} \cdot 1 = \frac{3}{5}$$

2. Observe that $-1 \le \sin x \le 1$ for all x. For x large (so we don't have to worry about negative numbers flipping our inequality), we have

$$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}.$$

Now $\lim_{x \to \infty} \frac{-1}{x} = \lim_{x \to \infty} \frac{1}{x} = 0$, so by the sandwich theorem, $\lim_{x \to \infty} \frac{\sin x}{x} = 0$.

3. Substitute $n = 1/h$. As $h \to 0^+$, $n \to \infty$.

$$\lim_{h \to 0^+} (1 + hx)^{1/h} = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

4. a.

$$2 \log x - 4 \log \frac{1}{y} - 3 \log(xy) = \log \frac{x^2}{(\frac{1}{y})^4 \ (xy)^3} = \log \frac{y}{x}$$

b.

$$2 \log x - 4 \log \frac{1}{y} - 3 \log(xy) = 2 \log x + 4 \log y - 3 \log x - 3 \log y = \log y - \log x = \log \frac{y}{x}$$

5. a. There are many possible answers. One is

$$\begin{aligned} p(x) &= 2x^3 - 17x + 1 \\ q(x) &= 5x^3 + x^2 - 8 \\ r(x) &= -x^4 + 17 \end{aligned}$$

b.

$$\lim_{x \to \infty} \frac{2x^3 - 17x + 1}{-x^4 + 17} = \lim_{x \to \infty} \frac{\frac{2}{x} - \frac{17}{x^3} + \frac{1}{x^4}}{-1 + \frac{17}{x^4}} = \frac{0 - 0 + 0}{-1 + 0} = 0$$

c.

$$\lim_{x \to \infty} \frac{2x^3 - 17x + 1}{5x^3 + x^2 - 8} = \lim_{x \to \infty} \frac{2 - \frac{17}{x^2} + \frac{1}{x^3}}{5 + \frac{1}{x} - \frac{8}{x^3}} = \frac{2 - 0 + 0}{5 + 0 - 0} = \frac{2}{5}$$

d.

$$\lim_{x \to \infty} \frac{-x^4 + 17}{5x^3 + x^2 - 8} = \lim_{x \to \infty} \frac{-x + \frac{17}{x^3}}{5 + \frac{1}{x} - \frac{8}{x^3}} = \text{“}\frac{-\infty - 0 + 0}{5 + 0 - 0}\text{”} = -\infty$$

e. We wish to consider an arbitrary rational function $f(x)$. Let m and n be the degrees of the numerator and the denominator and a and b be the leading coefficients—that is,

$$f(x) = \frac{ax^m + \text{lower order terms}}{bx^n + \text{lower order terms}}.$$

In general,

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{ax^m}{bx^n}. \qquad (*)$$

To prove this, we divide the top and bottom by x^n and consider several cases.

- If $m < n$, $\lim_{x \to \pm\infty} f(x) = 0$.
- If $m = n$, $\lim_{x \to \pm\infty} f(x) = \frac{a}{b}$.
- If $m > n$, the limit is $\pm\infty$, but we have to consider several subcases to say which:
  - If $\frac{a}{b}$ is positive, $\lim_{x \to \infty} f(x) = \infty$. If the numerator contains terms of degree greater than n but less than m, we should check that they do not change our answer. For example,

    $$\lim_{x \to \infty} \frac{x^5 - 1000x^4 + 1}{x^3 + 1} = \lim_{x \to \infty} \frac{x^2 - 1000x + \frac{1}{x^3}}{1 + \frac{1}{x^3}}.$$

    On the one hand, $x^2 \to \infty$, but $-1000x \to -\infty$, so when we put them together, what happens? In fact, the highest-degree term (x^2 in this case) will always win out, but showing this carefully would require some more work.
  - If $\frac{a}{b}$ is negative, $\lim_{x \to \infty} f(x) = -\infty$.
  - If $m > n$ and $m - n$ is even, $\lim_{x \to -\infty} f(x) = \infty$ or $-\infty$ according as $\frac{a}{b}$ is positive or negative. If $m - n$ is odd, the reverse is true.

In all cases, our general claim $(*)$ is true.