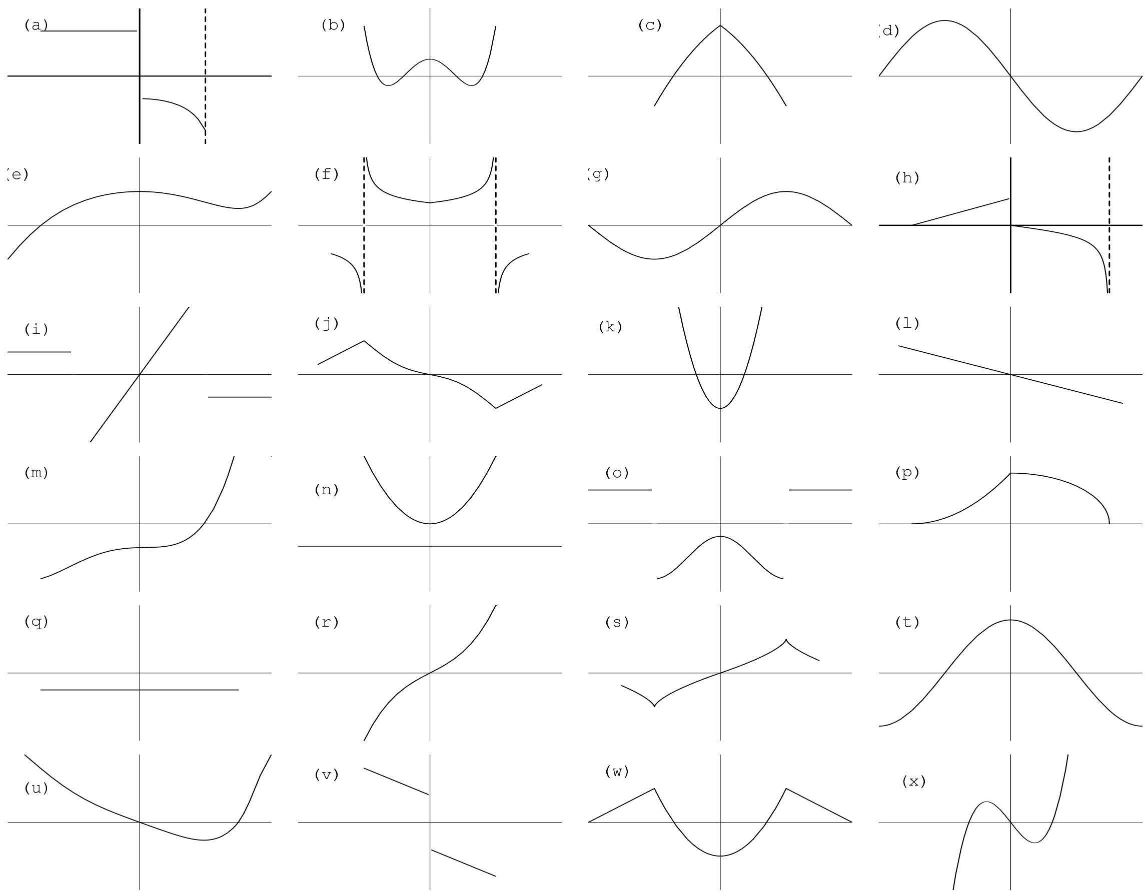
Problem Set 3

Tuesday, February 7

I. Problems to be graded on completion.

1. Below are the graphs of several functions, labeled (a) through (x). Some of these functions are the derivatives of others. For example, the derivative of w is i: where the tangent to w has positive slope, i is positive; where the tangent to w has negative slope, i is negative; where the tangent to w is horizontal, i is zero; where w has corners, that is, where the slope of w changes suddenly, i is discontinuous. Find the derivatives of b, c, e, g, h, j, l, p, r, s, t, u, and x.



2. Let $f(x) = \sqrt[3]{x}$. Find the derivative $f'(x)$ directly from the definition:

$$f'(x) = \lim_{u \to x} \frac{f(u) - f(x)}{u - x}.$$

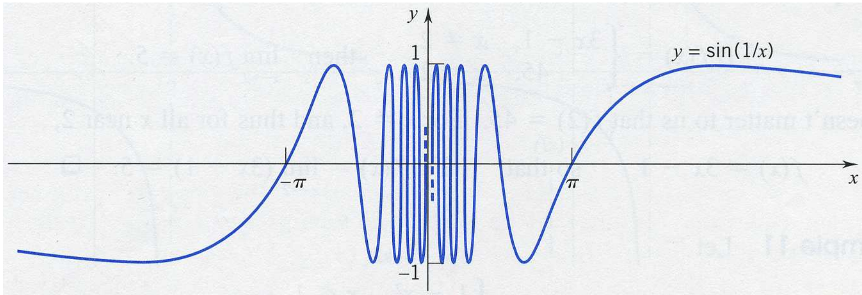
You may use the definition with $f(x+h)$ if you prefer, but it will be messier. Hint: If $f(x)$ were $\sqrt{x}$ rather than $\sqrt[3]{x}$, you would multiply the top and bottom by the conjugate $\sqrt{u} + \sqrt{x}$. The analogous thing here is $\sqrt[3]{u^2} + \sqrt[3]{ux} + \sqrt[3]{x^2}$. Second hint: Fractional exponents ($x^{1/3}$) will make your life easier.

3. Each of the following limits is the derivative of a function f at a number c. Find the limit as follows: determine f and c, find the derivative f' using the rules we know, and plug in $f'(c)$.
   a. $\lim_{h\to 0} \dfrac{\sqrt{4+h}-2}{h}$.
   b. $\lim_{h\to 0} \dfrac{(-2+h)^3+8}{h}$.
   c. $\lim_{h\to 0} \dfrac{\sin(\frac{\pi}{6}+h)-\frac{1}{2}}{h}$.
4. Differentiate (i.e. find the derivative of) the following functions using the power rule:
   a. $x^3 - 3x^2 + \frac{1}{2}x + 5$
   b. $x^{-3} - \frac{3}{x^2} + \pi^2$
   c. $2\sqrt[5]{x^4} - 7x^{9/7}$
   d. $x^4 + x^{-4} + x^{1/4} + x^{-1/4}$
5. Differentiate the following functions using the product rule:
   a. $(x^2-1)(x-3)$
   b. $3x^2(\frac{1}{x^3} - x)$
   c. $(2x+3)\ln x$
   d. $e^x \sin x - \tan x$
6. Differentiate the following functions using the chain rule:
   a. $(x^2+1)^2$
   b. $\frac{1}{x+1}$
   c. $\sin(e^x)$
   d. $\log \sqrt[4]{x^2+1}$
   e. $\cos(x^2) + (\cos x)^2$
7. Find $f'(0)$ given that $g(0) = 3$ and $g'(0) = 2$.
   a. $f(x) = xg(x)$
   b. $f(x) = 3x^2 g(x) - 5x$
   c. $f(x) = g(x) - \frac{1}{g(x)}$

II. Problems to be graded on correctness.

1. Differentiate
$$\tan\left(\frac{\cos\frac{1}{x}}{e^{3x^2-2}}\right).$$
   Hint: You would prefer not to use the quotient rule. Find a way to avoid that.
2. a. Differentiate $\sin(x+8)$ using the chain rule.
   b. Expand your answer to part (a) using an angle addition formula on the card in the back of your book. (You and I both know that $\cos 8 = -0.1455$, and neither of us cares. Leave it as $\cos 8$.)
   c. Expand $\sin(x+8)$ using an angle addition formula and *then* differentiate.
   d. Observe that your answers agree.
3. a. Differentiate $e^{\frac{1}{x}+\log x}$ using the chain rule.
   b. Simplify $e^{\frac{1}{x}+\log x}$ using the rules of exponents. Differentiate it using the product rule and the chain rule.
   c. Simplify both answers and make sure that they agree.

4. Recall that $\lim_{h\to 0} \sin(1/h)$ does not exist:



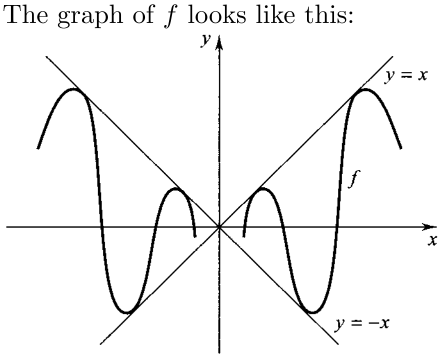
a. Show that $\lim_{h\to 0} h \sin(1/h) = 0$. Hint: How did you solve problem 2 last week?

b. Show that $\lim_{h\to 0} h^2 \sin(1/h) = 0$.

c. Let

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

The graph of f looks like this:

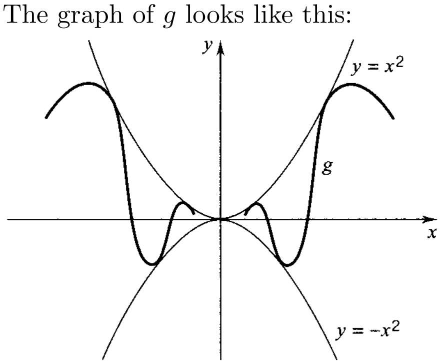


f is certainly continuous away from 0, and in part (a) you showed that it was continuous at 0 as well (do you understand why?). f is differentiable away from 0. Using the definition of the derivative (p. 107), show that f is *not* differentiable at 0.

d. Let

$$g(x) = \begin{cases} x^2\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

The graph of g looks like this:



g is continuous everywhere and differentiable away from 0. Show that g is also differentiable at 0.

e. Find $g'(x)$ for $x \neq 0$. From part (d), you also know $g'(0)$. Is $g'(x)$ continuous at 0?