Solutions to Problem Set 6

I. Problems to be graded on completion.

25. We have 180 feet of fence, so $2x + y + (y - 100) = 180$, so $y = 140 - x$. The area of the pen is $A = xy = 140x - x^2$, which is maximized when $0 = \frac{dA}{dx} = 140 - 2x$, so $x = 70$, so $y = 70$. But it doesn't make sense to have $y < 100$, so we should take $y = 100$ and $x = 40$.

31. If he drives at x miles per hour, it takes him $400/x$ hours to drive 400 miles, so the cost is $C = 400(25 + x/4) \cdot \frac{1}{100} + 12(400/x) = 10 + 1x + 4800x^{-1}$. Notice that we multiplied by $1/100$ since thd mileage cost is in cents, not dollars. This is minimized when $0 = \frac{dC}{dx} = 1 - 4800x^{-2}$, so $x = \sqrt{4800} = 40\sqrt{3} \approx 69.3$, but this is over the speed limit, so he should drive 55 mph.

11. The object hits the ground when $-\frac{2}{25}x^2 + x + 42 = 0$, so $x = -\frac{35}{2}$ or $x = 30$. If $x < 0$, the object is inside the cliff, so all our solutions must be between 0 and 30.

    It is always easier to minimize distance *squared*, so we do that. The square of the distance from the observer to the object is $r = (x - 2.6656)^2 + y^2 = (x - 2.6656)^2 + (-\frac{2}{25}x^2 + x + 42)^2$, which is minimized when $0 = \frac{dr}{dx} = 2(x - 2.6656) + 2(-\frac{2}{25}x^2 + x + 42)(-\frac{4}{25}x + 1) = \frac{16}{125}x^3 - \frac{12}{25}x^2 - \frac{236}{25}x + 78.6688$. Using a computer, $x = 28$, $x = 6.83$, and $x = -16.1$. We discard the negative solution since it is not between 0 and 30.

    When $x = 28$, the distance from the observer to the object is 26.4. When $x = 6.83$, the distance is 45.3. We should also see what happens at the beginning and end of the trajectory. When $x = 0$, the distance is 42.1. When $x = 30$, the distance is 27.3. Thus the minimum distance is 26.4 and the maximum is 45.3.

18. If we order 1000 ovens in batches of x ovens, we will have to order $1000/x$ batches. The problem tells us to assume that we have $x/2$ ovens in stock. Our costs are thus $C = 3(1000) + 200(1000/x) + 20(x/2) = 10x + 3000 + 200000x^{-1}$, which is minimized when $0 = \frac{dC}{dx} = 10 - 200000x^{-2}$, so $x = 100\sqrt{2} = 141.4$. We must order a whole number of microwaves. When $x = 141$, $C = 2828.44$, and when $x = 142$, $C = 2828.45$, so we should order in batches of 141.

46. Let $A(t)$ be the position of horse A at time t, $B(t)$ the position of horse B, $f(t) = A(t) - B(t)$ the distance between them, t_1 the time the beginning of the race, and t_2 the end. Since the horses start at the same point, $f(t_1) = 0$, and since they finish at the same time, the distance between them at the end is the same, so $f(t_2) = 0$. Thus by Rolle's theorem (or the mean value theorem) there is some time c in the middle of the race at which $f'(c) = 0$, so $A'(c) - B'(c) = 0$, so $A'(c) = B'(c)$.

1. Fix the slope of the line m, let x be the x-intercept, and let y be the y-intercept. Then $\dfrac{-5}{x-2} = m$, so $x = 2 - \dfrac{5}{m}$, and $\dfrac{5-y}{2} = m$, so $y = 5 - 2m$. The area of the triangle is $A = \frac{1}{2}xy = \frac{1}{2}(2 - 5m^{-1})(5 - 2m) = 10 - 2m - \frac{25}{2}m^{-1}$, which is minimzed when $0 = \frac{dA}{dm} = -2 + \frac{25}{2}m^{-2}$, so $m = \pm 5/2$. But the positive solution doesn't make sense, so $m = -5/2$.

2. Let x be the length of the edge of the square we're going to cut out. Then the height of the resulting box is x, the width is $8 - 2x$, and the length is $15 - 2x$, so the volume is $V = x(8 - 2x)(15 - 2x) = 4x^3 - 46x^2 + 120x$. This is maximized when $0 = \frac{dV}{dx} = 12x^2 - 92x + 120$, so $x = 6$ or $\frac{5}{3}$. But the rectangle was only 8 inches wide to begin with, so the biggest corner we can cut out is 4 inches, so $x = \frac{5}{3}$.

3. Since the cylinder is inscribed in the sphere, $r^2 + (\frac{1}{2}h)^2 = R^2$, so $r^2 = R^2 - \frac{1}{4}h^2$. The volume of the cylinder is $V = \pi r^2 h = \pi(R^2 - \frac{1}{4}h^2)h$, which is minimized when $0 = \frac{dV}{dh} = \pi(R^2 - \frac{1}{4}h^2) + \pi(-\frac{1}{2}h)h = \pi R^2 - \frac{3}{4}\pi h^2$, so $h = \frac{2}{\sqrt{3}}R$, so $r = \sqrt{\frac{2}{3}}R$.

4.
$$\lim_{x\to 0} \frac{\arctan 3x}{\arcsin x} = \lim_{x\to 0} \frac{\frac{1}{(3x)^2+1}\cdot 3}{\frac{1}{\sqrt{1-x^2}}} = 3$$

6.
$$\lim_{x\to 0} \frac{x^3 - 3x^2 + x}{x^3 - 2x} = \lim_{x\to 0} \frac{3x^2 - 6x + 1}{3x^2 - 2} = -\frac{1}{2}$$

8.
$$\lim_{x\to 1} \frac{2\log x}{x^2 - 1} = \lim_{x\to 1} \frac{2\frac{1}{x}}{2x} = 1$$

12.
$$\lim_{x\to 0^+} \frac{7^{\sqrt{x}} - 1}{2^{\sqrt{x}} - 1} = \lim_{x\to 0^+} \frac{e^{x^{1/2}\log 7} - 1}{e^{x^{1/2}\log 2} - 1} = \lim_{x\to 0^+} \frac{e^{x^{1/2}\log 7}\frac{1}{2}x^{-1/2}\log 7}{e^{x^{1/2}\log 2}\frac{1}{2}x^{-1/2}\log 2} = \lim_{x\to 0^+} \frac{e^{x^{1/2}\log 7}\log 7}{e^{x^{1/2}\log 2}\log 2} = \frac{\log 7}{\log 2}$$

2.
$$\lim_{x\to\infty} \frac{(\log x)^2}{2^x} = \lim_{x\to\infty} \frac{2\log x\frac{1}{x}}{2^x\log 2} = \lim_{x\to\infty} \frac{2\log x}{x2^x\log 2}$$
$$= \lim_{x\to\infty} \frac{2\frac{1}{x}}{2^x\log 2 + x2^x(\log 2)^2} = \lim_{x\to\infty} \frac{2}{x2^x\log 2 + x^2 2^x(\log 2)^2} = \frac{2}{\infty} = 0$$

8.
$$\lim_{x\to(1/2)^-} \frac{2\log(4-8x)}{\tan \pi x} = \lim_{x\to(1/2)^-} \frac{2\frac{-8}{4-8x}}{\pi(\sec\pi x)^2}$$
$$= \lim_{x\to(1/2)^-} \frac{-16}{\pi}\frac{(\cos\pi x)^2}{4-8x} = \lim_{x\to(1/2)^-} \frac{-16}{\pi}\frac{-2\pi\cos\pi x\sin\pi x}{-8} = 0$$

28.
$$\lim_{x\to 0} e^{\left(\frac{1}{x}\log(\cos x - \sin x)\right)} = e^{\left(\lim_{x\to 0}\frac{\log(\cos x - \sin x)}{x}\right)} = e^{\left(\lim_{x\to 0}\frac{\frac{-\sin x - \cos x}{\cos x - \sin x}}{1}\right)} = e^{-1} = \frac{1}{e}$$

32.
$$\lim_{x\to 1}\left(\frac{1}{x-1} - \frac{x}{\log x}\right) = \lim_{x\to 1}\frac{\log x - x(x-1)}{(x-1)\log x} = \lim_{x\to 1}\frac{\log x - x^2 + x}{(x-1)\log x}$$
$$= \lim_{x\to 1}\frac{x^{-1} - 2x + 1}{\log x + (x-1)x^{-1}} = \lim_{x\to 1}\frac{x^{-1} - 2x + 1}{\log x + 1 - x^{-1}} = \lim_{x\to 1}\frac{-x^{-2} - 2}{x^{-1} + x^{-2}} = -\frac{3}{2}$$

II. Problems to be graded on correctness.

1. It is convenient to solve the problem in general. Let a be the price per km underground and b be the price underwater. Let x be the distance from the point where the line leaves the shore to the point on the shore nearest to the island. Then the distance from the power plant to the point where the line leaves the shore is $4-x$, and the distance from there to the island is $\sqrt{x^2+1}$, so the cost is $C = a(4-x) + b\sqrt{x^2+1}$, which is minimized when $0 = \frac{dC}{dx} = -a + b\dfrac{x}{\sqrt{x^2+1}}$. Notice that the distance 4 km is gone at this point. After some algebra, $x = \dfrac{a}{\sqrt{b^2-a^2}}$.
   a. $x = \dfrac{30000}{\sqrt{50000^2 - 30000^2}} = \dfrac{3}{4}$, so the cost is 160000.
   b. If it is cheaper to lay cable underwater, clearly we should go straight from the power plant to the island, so $x = 4$, so the cost is $30000\sqrt{17} \approx 123693.17$. But the calculus solution gives us the square root of a negative number, which does not make sense.
   c. If $a = 30000$, $x = 2$, and $x = \dfrac{a}{\sqrt{b^2-a^2}}$, then $b = 15000\sqrt{5} \approx 33541.02$.
2. Nick wants to figure out how many homework problems to assign. He wants to maximize the number of problems times the number of students that are willing to do that many problems, that is, the total number of problems done by his students. Suppose that he has S students and that and that for each problem he assigns, R students will refuse to do the assignment. How many problems should he assign?
3. If he assigns x problems, $S-Rx$ students will do the assignment. We wish to maximize $x(S-Rx) = Sx - Rx^2$, so $S - 2Rx = 0$, so $x = \dfrac{S}{2R}$. Notice that at that point, half the class will refuse to do the assignment, so maybe our model is not realistic.
4. a. Since $|f(x) - f(y)| \leq M|x-y|$ for all x and y, $\dfrac{|f(x)-f(y)|}{|x-y|} \leq M$ for all $x \neq y$, so $\lim\limits_{x\to y} \dfrac{|f(x)-f(y)|}{|x-y|} \leq M$, but

$$\lim_{x\to y} \frac{|f(x)-f(y)|}{|x-y|} = \lim_{x\to y} \left|\frac{f(x)-f(y)}{x-y}\right| = \left|\lim_{x\to y} \frac{f(x)-f(y)}{x-y}\right| = |f'(y)|,$$

   so $|f'(y)| \leq M$ for all y.

   b. Assume that there are two numbers x and y for which $|f(x)-f(y)| > M|x-y|$, so $\left|\dfrac{f(x)-f(y)}{x-y}\right| > M$. By the mean value theorem, there is a c between x and y such that $f'(c) = \dfrac{f(x)-f(y)}{x-y}$, so $|f'(c)| > M$. But this contradicts our hypothesis that $|f'(x)| \leq M$ for all x, so assumption that there are x and y for which $|f(x)-f(y)| > M|x-y|$ must have been false, so $|f(x)-f(y)| \leq M|x-y|$ for all x and y.
5. Observe that as $h \to 0$, $h^2 \to 0^+$, so $1/h^2 \to \infty$, so $e^{1/h^2} \to \infty$ and $e^{-1/h^2} \to e^{-\infty} = 0$.
   a.

$$\lim_{h\to 0} \frac{e^{-h^{-2}}}{h} = \lim_{h\to 0} \frac{e^{-h^{-2}}(2h^{-3})}{1} = \lim_{h\to 0} 2\frac{e^{-h^{-2}}}{h^3} = \lim_{h\to 0} 2\frac{e^{-h^{-2}}(2h^{-3})}{3h^2} = \lim_{h\to 0} \frac{4}{3}\frac{e^{-h^{-2}}}{h^5}$$

   so as we use L'Hôpital's rule repeatedly, power of h in the denominator gets higher and higher, so the problem is getting harder, not easier.

   b.

$$\lim_{h\to 0} \frac{e^{-h^{-2}}}{h} = \lim_{h\to 0} \frac{h^{-1}}{e^{h^{-2}}} = \lim_{h\to 0} \frac{-h^{-2}}{e^{h^{-2}}(-2h^{-3})} = \lim_{h\to 0} \frac{1}{2}\frac{h}{e^{h^{-2}}} = \frac{1}{2}\frac{0}{\infty} = 0$$

   c.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{e^{-1/h^2} - 0}{h} = 0$$