Problem Set 8
March 28 and 29, in class

Recall some facts:

• If \( f(x) \leq g(x) \) for all \( x \) in \([a, b]\) then \( \int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx \).

• If \( k \) is a constant then \( \int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx \).

• \( \int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \).

1. (a) Understand the following proof:

\[
-|f(x)| \leq f(x) \leq |f(x)|
\]
\[
- \int_a^b |f(x)| \, dx \leq \int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx
\]
\[
\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx
\]

(b) Explain why the last inequality makes sense. Hint: Find some functions for which the left-hand side and the right-hand side are not equal. Then explain why the left-hand side always has to be less.

2. Suppose that \( f \) is continuous, \( a < b \), and \( \int_a^b f(x) \, dx = 0 \).

(a) Does it necessarily follow that \( f(x) = 0 \) for all \( x \) in \([a, b]\)?

(b) Does it necessarily follow that \( f(x) = 0 \) for some \( x \) in \([a, b]\)?

(c) Does it necessarily follow that \( \int_a^b |f(x)| \, dx = 0 \)?

(d) Does it necessarily follow that \( \left| \int_a^b f(x) \, dx \right| = 0 \)?

(e) Must all the upper sums \( U(P) \) be nonnegative?

(f) Must all the upper sums \( U(P) \) be positive?

(g) Can a lower sum \( L(P) \) be positive?
3. Suppose that \( f \) and \( g \) are continuous, \( a < b \), and \( \int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx \).

(a) Does it necessarily follow that \( \int_a^b [f(x) - g(x)] \, dx \geq 0 \)?

(b) Does it necessarily follow that \( f(x) \geq g(x) \) for all \( x \) in \([a, b]\)?

(c) Does it necessarily follow that \( f(x) \geq g(x) \) for some \( x \) in \([a, b]\)?

(d) Does it necessarily follow that \( \left| \int_a^b f(x) \, dx \right| \geq \int_a^b |g(x)| \, dx \)?

(e) Does it necessarily follow that \( \int_a^b |f(x)| \, dx \geq \int_a^b |g(x)| \, dx \)?

(f) Does it necessarily follow that \( \int_a^b |f(x)| \, dx \geq \int_a^b g(x) \, dx \)?

4. While \( \int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \), in general it is not true that

\[
\int_a^b f(x)g(x) \, dx = \left( \int_a^b f(x) \, dx \right) \left( \int_a^b g(x) \, dx \right).
\]

(a) Find two numbers \( a \) and \( b \) and two functions \( f \) and \( g \) for which the equation above fails—any random choice of \( a, b, f, \) and \( g \) should work.

(b) Find two numbers \( a \) and \( b \) and two functions \( f \) and \( g \) for which the equation above holds—you will have to choose \( a, b, f, \) and \( g \) carefully.