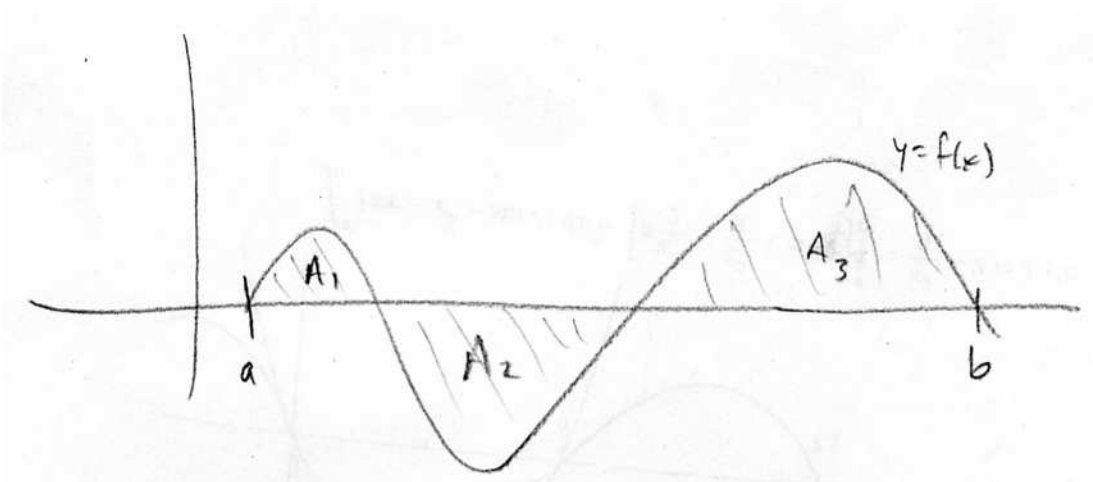
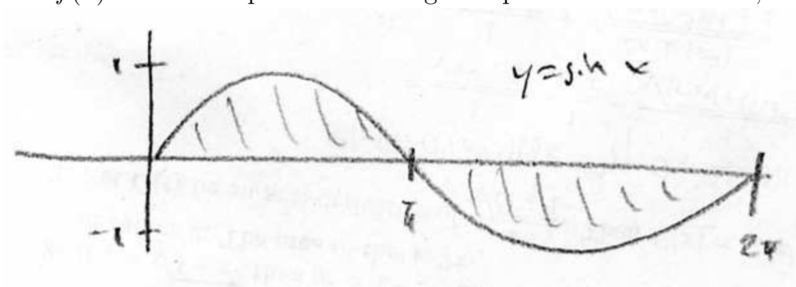
Solutions to Problem Set 8

1. (a) To go from the second to the third lines, observe that for any numbers m and n, the inequality $|m| \le n$ is equivalent to the inequality $-n \le m \le n$. If $-n \le m \le n$, then $-n \le m$, so $n \ge -m$, so $-m \le n$, and also $m \le n$, so $|m| \le n$.

   (b) If $f(x)$ is always positive or always negative, then $\left|\int_a^b f(x)\,dx\right| = \int_a^b |f(x)|\,dx$. But if $f(x)$ is sometimes positive and sometimes negative, then the positive and negative parts might cancel out on the left-hand side, but on the right-hand side everything will add up. For example, in the figure below, $\left|\int_a^b f(x)\,dx\right| = A_1 - A_2 + A_3$, whereas $\int_a^b |f(x)|\,dx = A_1 + A_2 + A_3$.



2. (a) No. $f(x)$ could have positive and negative pieces that cancel out, as in the figure below.



   Specifically, we could take $a = 0$, $b = 2\pi$, and $f(x) = \sin x$.

   (b) Yes. If $f(x)$ were never zero, then it would either be positive always or negative always (since it is continuous, it cannot jump), so $\int_a^b f(x)\,dx$ could not be 0.

   (c) No. We can reuse our counterexample from part (a).

   (d) Yes. The absolute value of 0 is 0.

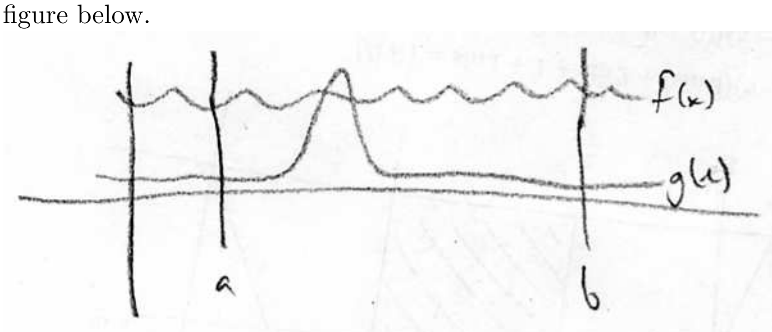
(e) Yes. $\int_a^b f(x)\ dx$ is less than or equal to all the upper sums by definition, so they must all be greater than or equal to 0.

(f) No. Take $f(x) = 0$; then all the upper sums are 0.

(g) No. $\int_a^b f(x)\ dx$ is greater than or equal to all the lower sums by definition, so they must all be less than or equal to 0.

3. (a) Yes. $\int_a^b [f(x) - g(x)]\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx \geq 0.$

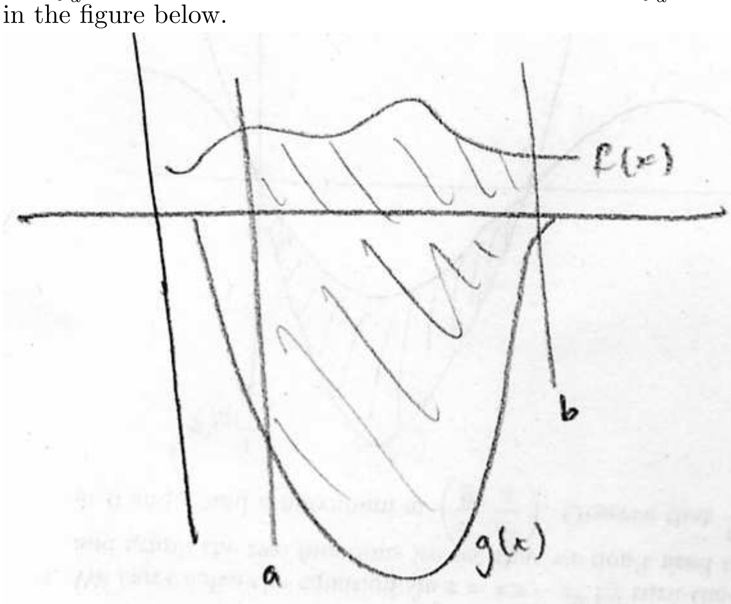
(b) No. $g(x)$ could jump up above $f(x)$ for a little while but be small most of the time, as in the figure below.



Specifically, we could take $a = 0$, $b = 1$, $f(x) = 2$, and $g(x) = 3x$. Then $\int_a^b f(x)\ dx = 2$ and $\int_a^b g(x)\ dx = \frac{3}{2}$, but $f(1) < g(1)$.

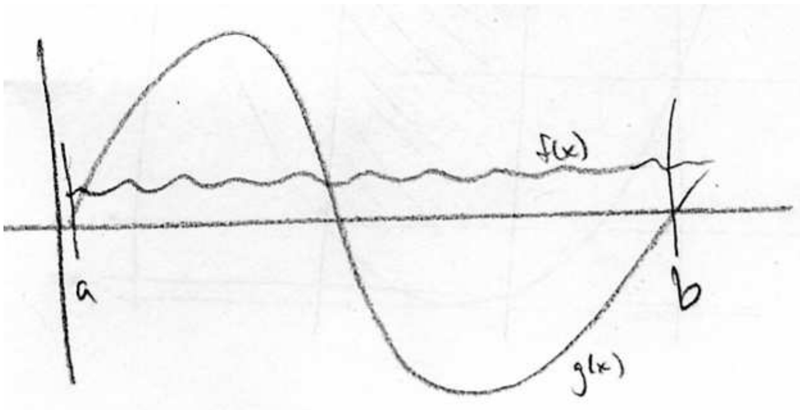
(c) Yes. If it were not true that $f(x) \geq g(x)$ for some x in $[a, b]$, then we would have $f(x) < g(x)$ for *all* x in $[a, b]$, so $\int_a^b f(x)\ dx < \int_a^b g(x)\ dx$.

(d) No. $\int_a^b f(x)\ dx$ could be a small positive number and $\int_a^b g(x)\ dx$ is a large negative number, as in the figure below.



Specifically, we could take $a = 0$, $b = 1$, $f(x) = 1$, and $g(x) = -100$.

(e) No. $f(x)$ could be small and positive, while $g(x)$ could have large positive and large negative pieces that cancel out, as in the figure below.



Specifically, we could take $a = 0$, $b = 2\pi$, $f(x) = 1$, and $g(x) = 10 \sin x$. Then $\int_a^b f(x)\ dx = 2\pi$ and $\int_a^b g(x)\ dx = 0$, but $\int_a^b |g(x)|\ dx = 40$.

(f) Yes. $\int_a^b g(x)\ dx \le \int_a^b f(x)\ dx \le \left| \int_a^b f(x)\ dx \right| \le \int_a^b |f(x)|\ dx$. The last inequality was proved in problem 1.

4. (a) Let $a = 0$, $b = 1$, $f(x) = x$, and $g(x) = x^2$. Then $\int_0^1 x\ dx = \frac{1}{2}$ and $\int_0^1 x^2\ dx = \frac{1}{3}$, but $\int_0^1 x^3\ dx = \frac{1}{4}$, which is different from $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$.

(b) Let $a = 0$, $b = \frac{3}{2}$, $f(x) = x$, and $g(x) = x^2$. Then $\int_0^{3/2} x\ dx = \frac{9}{8}$, $\int_0^{3/2} x^2\ dx = \frac{9}{8}$, and $\int_0^{3/2} x^3\ dx = \frac{81}{64}$.