Solutions to Problem Set 8

1. (a) To go from the second to the third lines, observe that for any numbers $m$ and $n$, the inequality $|m| \leq n$ is equivalent to the inequality $-n \leq m \leq n$. If $-n \leq m \leq n$, then $n \geq -m$, so $-m \leq n$, and also $m \leq n$, so $|m| \leq n$.

(b) If $f(x)$ is always positive or always negative, then $\left| \int_a^b f(x) \, dx \right| = \int_a^b |f(x)| \, dx$. But if $f(x)$ is sometimes positive and sometimes negative, then the positive and negative parts might cancel out on the left-hand side, but on the right-hand side everything will add up. For example, in the figure below, $\left| \int_a^b f(x) \, dx \right| = A_1 - A_2 + A_3$, whereas $\int_a^b |f(x)| \, dx = A_1 + A_2 + A_3$.

2. (a) No. $f(x)$ could have positive and negative pieces that cancel out, as in the figure below.

Specifically, we could take $a = 0$, $b = 2\pi$, and $f(x) = \sin x$.

(b) Yes. If $f(x)$ were never zero, then it would either be positive always or negative always (since it is continuous, it cannot jump), so $\int_a^b f(x) \, dx$ could not be 0.

(c) No. We can reuse our counterexample from part (a).

(d) Yes. The absolute value of 0 is 0.
(e) Yes. \( \int_a^b f(x) \, dx \) is less than or equal to all the upper sums by definition, so they must all be greater than or equal to 0.

(f) No. Take \( f(x) = 0 \); then all the upper sums are 0.

(g) No. \( \int_a^b f(x) \, dx \) is greater than or equal to all the lower sums by definition, so they must all be less than or equal to 0.

3. (a) Yes. 
\[
\int_a^b [f(x) - g(x)] \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx \geq 0.
\]

(b) No. \( g(x) \) could jump up above \( f(x) \) for a little while but be small most of the time, as in the figure below.

![Graph showing function and its integral](image)

Specifically, we could take \( a = 0, b = 1, f(x) = 2, \) and \( g(x) = 3x \). Then \( \int_a^b f(x) \, dx = 2 \) and \( \int_a^b g(x) \, dx = \frac{3}{2} \), but \( f(1) < g(1) \).

(c) Yes. If it were not true that \( f(x) \geq g(x) \) for some \( x \) in \( [a, b] \), then we would have \( f(x) < g(x) \) for all \( x \) in \( [a, b] \), so \( \int_a^b f(x) \, dx < \int_a^b g(x) \, dx \).

(d) No. \( \int_a^b f(x) \, dx \) could be a small positive number and \( \int_a^b g(x) \, dx \) is a large negative number, as in the figure below.

![Graph showing function and its integral](image)

Specifically, we could take \( a = 0, b = 1, f(x) = 1, \) and \( g(x) = -100 \).
(e) No. \( f(x) \) could be small and positive, while \( g(x) \) could have large positive and large negative pieces that cancel out, as in the figure below.

![Graph of f and g functions](image)

Specifically, we could take \( a = 0 \), \( b = 2\pi \), \( f(x) = 1 \), and \( g(x) = 10 \sin x \). Then \( \int_a^b f(x) \, dx = 2\pi \) and \( \int_a^b g(x) \, dx = 0 \), but \( \int_a^b |g(x)| \, dx = 40 \).

(f) Yes. \( \int_a^b g(x) \, dx \leq \int_a^b f(x) \, dx \leq \left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx \). The last inequality was proved in problem 1.

4. (a) Let \( a = 0 \), \( b = 1 \), \( f(x) = x \), and \( g(x) = x^2 \). Then \( \int_0^1 x \, dx = \frac{1}{2} \) and \( \int_0^1 x^2 \, dx = \frac{1}{3} \), but
\[
\int_0^1 x^3 \, dx = \frac{1}{4},
\]
which is different from \( \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6} \).

(b) Let \( a = 0 \), \( b = \frac{3}{2} \), \( f(x) = x \), and \( g(x) = x^2 \). Then \( \int_0^{3/2} x \, dx = \frac{9}{8} \), \( \int_0^{3/2} x^2 \, dx = \frac{9}{8} \), and
\[
\int_0^{3/2} x^3 \, dx = \frac{81}{64}.
\]