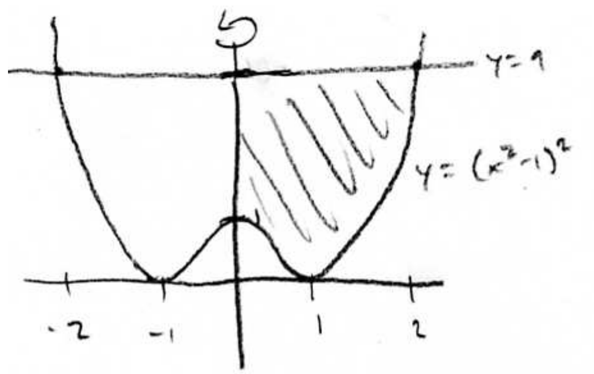
Solutions to Practice Exam 3

1. (a) $y = (x^2 - 1)^2$ has zeros when $x = \pm 1$ and critical points (i.e. mins and maxes) when $0 = y' = 4x(x^2 - 1)$, so $x = 0$ or ± 1. It intersects the line $y = 9$ when $x^2 - 1 = \pm 3$, but we can ignore the negative root since $x^2 - 1 \geq -1$, so $x^2 = 4$, so $x = \pm 2$.

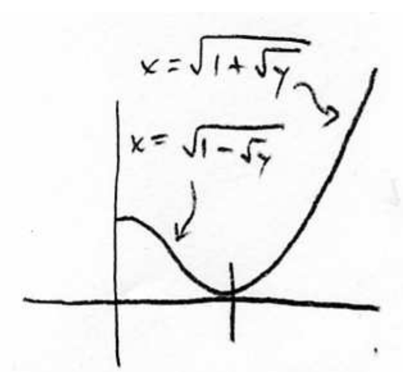


(b) A typical shell has radius x and height $9 - (x^2 - 1)^2$, so

$$\int_0^2 2\pi x[9 - (x^2 - 1)^2]\ dx = 9\pi \int_0^2 2x\ dx - \pi \int_0^2 2x(x^2 - 1)^2\ dx$$
$$= 9\pi x^2 \Big|_0^2 - \pi \int_{-1}^3 u^2\ du = 36\pi - \pi \frac{u^3}{3}\Big|_{-1}^3 = 36\pi - \frac{\pi}{3}(27 - (-1)) = 27\pi - \frac{\pi}{3}$$

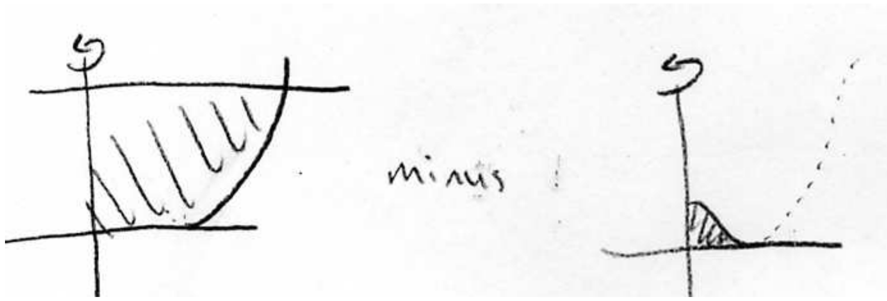
where we substituted $u = x^2 - 1$.

(c) We solve for x: $y = (x^2 - 1)^2$, so $x^2 - 1 = \pm\sqrt{y}$, so $x^2 = 1 \pm \sqrt{y}$, so $x = \pm\sqrt{1 \pm \sqrt{y}}$, where the two plus-or-minuses are independent. The outside plus-or-minus just gives us the mirror image across the y-axis, but the inside one distinguishes these two pieces of the curve:



Observe that $x = \sqrt{1 + \sqrt{y}}$ is defined for all $y \geq 0$, while $x = \sqrt{1 - \sqrt{y}}$ is not defined for $y > 1$, which agrees with the picture.

We should rotate $x = \sqrt{1+\sqrt{y}}$ and $x = \sqrt{1-\sqrt{y}}$ separately and subtract the smaller volume from the larger.



For $x = \sqrt{1+\sqrt{y}}$, the volume of a typical disk is $\pi(1+\sqrt{y})\ dy$, so

$$\int_0^9 \pi(1+\sqrt{y})\ dy = \pi\left[y + \frac{y^{3/2}}{3/2}\right]_0^9 = \pi\left(9 + \frac{27}{3/2}\right) = 27\pi.$$

For $x = \sqrt{1-\sqrt{y}}$, the volume of a typical disk is $\pi(1-\sqrt{y})\ dy$, so

$$\int_0^1 \pi(1-\sqrt{y})\ dy = \pi\left[y - \frac{y^{3/2}}{3/2}\right]_0^1 = \pi\left(1 - \frac{2}{3}\right) = \frac{\pi}{3}.$$

The volume of the solid is $27\pi - \frac{\pi}{3}$, as before.

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2. Let t be the time in minutes and w be amount of water in the bathtub in gallons. When $t = 0$, $w = 200$. When $t = 5$, $w = 160$.

(a) $\dfrac{dw}{dt} = kw$, so $\dfrac{dw}{w} = k\ dt$, so $\log w = kt + C$, so $w = e^{kt+C} = e^C e^{kt}$. Using the intial conditions, $e^C = 200$ and $k = \frac{1}{5}\log\frac{200}{160} = \frac{1}{5}\log\frac{4}{5}$, so

$$w = 200e^{\frac{1}{5}\log\frac{4}{5}\cdot t} = 200\left(\frac{4}{5}\right)^{t/5}.$$

(b) $\dfrac{dw}{dt} = kwt$, so $\dfrac{dw}{w} = kt\ dt$, so $\log w = \frac{1}{2}kt^2 + C$, so $w = e^{\frac{1}{2}kt^2+C} = e^C e^{kt^2/2}$. Using the initial conditions, $e^C = 200$ and $k/2 = \frac{1}{25}\log\frac{160}{200} = \frac{1}{25}\log\frac{4}{5}$, so

$$w = 200e^{\frac{1}{25}\log\frac{4}{5}\cdot t^2} = 200\left(\frac{4}{5}\right)^{(t/5)^2}.$$

3. (a) The integrating factor is $e^{\int -2/x \ dx} = e^{-2\log x} = x^{-2}$.

$$\begin{aligned}
y' - 2x^{-1}y &= x^{-1} \\
x^{-2}y' - 2x^{-3}y &= x^{-3} \\
(x^{-2}y)' &= x^{-3} \\
x^{-2}y &= \int x^{-3} \ dx = \frac{x^{-2}}{-2} + C \\
y &= -\frac{1}{2} + Cx^2.
\end{aligned}$$

Using the initial condition, $5/2 = -1/2 + C$, so $C = 3$, so $y = 3x^2 - \frac{1}{2}$.

(b)

$$\begin{aligned}
\frac{dy}{dx} - \frac{2y}{x} &= \frac{1}{x} \\
\frac{dy}{dx} &= \frac{2y}{x} + \frac{1}{x} = \frac{2y+1}{x} \\
\frac{dy}{2y+1} &= \frac{dx}{x} \\
\int \frac{dy}{2y+1} &= \int \frac{dx}{x} \\
\frac{1}{2}\log(2y+1) &= \log x + C
\end{aligned}$$

where on the left-hand side we substituted $u = 2y + 1$. Using the intial condition, $C = \frac{1}{2}\log 6$, so

$$\begin{aligned}
\frac{1}{2}\log(2y+1) &= \log x + \frac{1}{2}\log 6 \\
\log(2y+1) &= 2\log x + \log 6 = \log(6x^2) \\
2y + 1 &= 6x^2 \\
y &= 3x^2 - \frac{1}{2}
\end{aligned}$$

which agrees with part (a).

(c)

$$y' - \frac{2}{x}y = (6x) - \frac{2}{x}\left(3x^2 - \frac{1}{2}\right) = 6x - \left(6x - \frac{1}{x}\right) = \frac{1}{x}.$$