Sandbag Problem

Problem. A hot-air balloon is moving straight up at a rate of 4 feet per second. You are standing 100 feet from the spot on the ground directly below the balloon. When the balloon is 50 feet in the air, the balloonist drops a sandbag. At what rate is the angle of inclination of your line of sight to the sandbag decreasing when the sandbag is halfway to the ground?

Solution. Let the moment when the balloonist drops the sandbag be $t = 0$. The bag's initial position is 50 feet above the ground. It is dropped by the balloonist—that is, it is initially at rest with respect to the balloon—so its initial velocity *with respect to the ground* is 4 feet per second upward. Thus its height y is described by the equation

$$y = -16t^2 + 4t + 50.$$

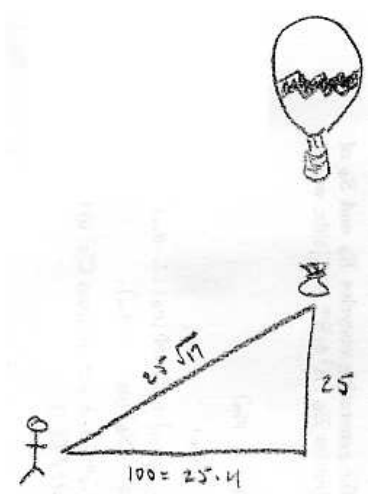
Let θ be the angle of inclination of your line of sight to the sandbag. Then

$$\tan\theta = \frac{y}{100}.$$

Taking the derivative of both sides with respect to time,

$$\begin{aligned}(\sec\theta)^2\frac{d\theta}{dt} &= \frac{1}{100}\frac{dy}{dt}\\ \frac{d\theta}{dt} &= \frac{1}{100}\frac{dy}{dt}(\cos\theta)^2.\end{aligned}$$

First we wish to find $(\cos\theta)^2$. When the bag is halfway down, $y = 25$, so we have the following right triangle:



Thus $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{25\cdot 4}{25\sqrt{17}} = \frac{4}{\sqrt{17}}$, so $(\cos\theta)^2 = \frac{16}{17}$.

To find $\frac{dy}{dt}$, we first find t when $y = 25$.

$$\begin{aligned} 25 &= -16t^2 + 4t + 50 \\ 0 &= -16t^2 + 4t + 25 \\ t &= \frac{-4 \pm \sqrt{4^2 - 4(-16)(25)}}{2(-16)} \\ &= \frac{1 \mp \sqrt{101}}{8}. \end{aligned}$$

We choose the positive square root so that time in questions is after the bag has been dropped. When $t = \frac{1+\sqrt{101}}{8}$,

$$\frac{dy}{dt} = -32t + 4 = -4\sqrt{101}.$$

Putting this all together,

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{1}{100}\frac{dy}{dt}(\cos\theta)^2 \\ &= \frac{1}{100}(-4\sqrt{101})\frac{16}{17} \\ &= \frac{-16\sqrt{101}}{425}\frac{\text{rad}}{\text{sec}} = \frac{-576\sqrt{101}}{85\pi}\frac{\text{deg}}{\text{sec}} \approx -21.6\ \frac{\text{deg}}{\text{sec}}. \end{aligned}$$